PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Storm Surges on a Continental Shelf

N. S. Heaps

Phil. Trans. R. Soc. Lond. A 1965 **257**, 351-383
doi: 10.1098/rsta.1965.0009

STORM SURGES ON A CONTINENTAL SHELF

By N. S. HEAPS

The University of Liverpool Tidal Institute and Observatory

(*Communicated by J. Proudman, F.R.S.—Received* 11 *March* 1964)

CONTENTS

This paper gives a theoretical analysis of storm surges generated on a continental shelf by moving wind fields; surges generated in the ocean and propagated on to the shelf are also investigated.

A continental shelf of uniform depth and width, bounded by a long straight coast and connected with an infinitely deep ocean, is considered. In studying the motion of the shelf sea, bottom friction and Coriolis force are taken into account, and it is supposed that conditions are the same in all vertical sections normal to the coast.

The theory is applied to the problem of storm surges on the west coast of the British Isles. It is shown that these surges are generated by wind fields, associated with moving depressions, acting over the sea of the continental shelf to the south of Ireland. In two cases the observed surge at Milford Haven has been reproduced satisfactorily by the theory.

1. Introduction

In a recent investigation, Lennon (1963) has identified weather conditions associated with the generation of major storm surges along the west coast of the British Isles. It appears that large surges occur at Avonmouth and Liverpool when the wind fields of moving depressions sweep across the sea of the continental shelf, at a critical speed, towards the coast. In the present paper an attempt is made to gain more knowledge of the mechanism of the surge generation by considering a simple mathematical model of the continental shelf, investigating the dynamic response of the water on it to wind stress fields with moving boundaries. The effects of bottom friction and the Coriolis force are taken into account and conditions are assumed to be the same in all planes normal to the coastline which is straight. The idealized shelf is of uniform depth and width and is connected with an infinitely deep ocean.

Some of the depressions mentioned above progress from the ocean on to the continental shelf and one may therefore suppose that part of the surge at the coast is due to disturbances generated meteorologically in the ocean and propagated on to the shelf. This aspect of the problem is examined using the mathematical model by determining the elevation of the sea surface at the coast due to a prescribed time-variation in the elevation at the oceanic edge of the shelf, the latter being chosen in the form of a half sine wave to represent an incoming disturbance. No method is given for the estimation of the elevation at the edge of the continental shelf in terms of meteorological conditions over the ocean.

The basic wind field defined in the present work covers an area of the shelf sea between two lines parallel to the coast: the wind stress is constant and uniformly distributed over this area, having both onshore and longshore components. The variation in elevation of the sea surface at the coast due to such a field being suddenly created and then maintained is determined. Fields of this type may be superimposed (with appropriate time lags between the commencement of each) and the coastal elevation obtained due to a wind region which is created at the oceanic edge of the shelf and subsequently moves onshore.

Two examples of storm surges which have occurred along the west coast of the British Isles are considered; in each case the observed surge at Milford Haven is reproduced satisfactorily by the theory. In this work, the effect of wind fields, which act over a sea area to the south of Ireland, is determined. It is assumed that the level of the sea surface responds to changes in atmospheric pressure according to the statical law (Charnock & Crease 1957).

2. Previous work

The published work on storm surges has been summarized by Charnock & Crease (1957), Groen & Groves (1962), Lauwerier (1962), and Welander (1961). Here we mention only those theoretical papers which may be regarded as having a direct bearing on the present investigation.

Lauwerier (1957 *a*, *b*) considered a shelf sea, as defined in the present paper, and calculated the response of the water level at the coast to several time-dependent uniform wind stress fields covering the entire sea area; both the effect of the Earth's rotation and bottom friction were taken into account. Ignoring bottom friction, Kajiura (1959) also dealt analytically with the same type of sea and determined water-level changes due to a wind field in the form of simple progressive waves moving parallel to the shore, and a field commencing suddenly, rotating in direction, uniform over the shelf at any instant. Reid (1956) described a graphical procedure for evaluating the response of the sea level at the shore of a sloping shelf to a wind field moving directly onshore: neglecting the Coriolis force and friction, one-dimensional hydrodynamic equations were solved by the method of characteristics. Storm surges due to a model hurricane, moving on a simple continental shelf of constant depth and width, were investigated theoretically by Ichiye (1962). The equations of motion were simplified as discussed by Freeman, Baer & Jung (1957) to yield a first-order approximation to the elevation of the sea surface; from this, a second-order approximation was deduced.

Several authors have studied analytically the influence of wind fields upon infinite and semi-infinite seas, usually of uniform depth. Lauwerier (1955) considered the motion of a half-plane sea subjected to a uniform time-dependent wind field of constant direction, including both the Coriolis and frictional parameters. Crease (1956) determined the surface elevation and horizontal flow due to a stationary force suddenly applied and maintained over one half of an infinite rotating sea with no bottom friction; the elevation was also obtained in the case when the edge of the generating area moved forward with a constant velocity and the effect of a coastal barrier was discussed. Takegami (1936) investigated the response of the sea surface of an unbounded ocean, and of an ocean bounded by a straight coast, to travelling wind regions; the problem was treated in two dimensions (vertical and horizontal directions) without consideration of the Earth's rotation and assuming that there was no bottom friction. Theoretical patterns of the sea-level variations in an open sea and in some typical models of bounded seas where the meteorological disturbances propagate with a constant speed and there are the effects of Coriolis parameter and bottom friction, have been obtained by Miyazaki (1952, 1956). In this analysis the travelling atmospheric disturbances are of the general form $f(x-Vt, y)$ where x, y are horizontal co-ordinates, V is the speed of propagation, and t the time. Long waves in a sea of infinite horizontal extent, caused by circular wind fields of various types, have been studied by Ichiye (1950), while Kajiura (1956) has determined the forced wave generated by a travelling circular atmospheric disturbance in deep water.

The effect of wind, air pressure gradients, and density gradients on the sea level and the currents near a straight coast have been investigated by Nomitsu & Takegami (1934), and Nomitsu (1934). Various bottom conditions and the Coriolis force were taken into account. In the first of these papers the steady state conditions maintained by wind all over the surface of a semi-infinite ocean bounded by a long straight coast were determined; the case of an enclosed sea was also considered. In the second paper, non-stationary conditions brought about by a constant wind which begins to blow suddenly over a sea bounded by two parallel coasts were studied. Nomitsu (1935) extended the work for its application to the problem of the storm surge along the coast of a very shallow sea or in a long narrow bay, neglecting the rotation of the Earth. The steady-state value of surface elevation was obtained without the usual assumption that the elevation is negligibly small compared with the total depth of the water. The changing state in the development of a surge was dealt with by considering sea-level variations in a uniform canal of finite length caused by a time-dependent wind (or barometric gradient) over the water surface. The effect on sea level of the movement of a meteorological disturbance was investigated by considering the case of a force in the form of a progressive wave acting over an endless canal.

Comparatively little work appears to have been published on the theory of external surges, which is concerned with the propagation of free waves into a sea across an open boundary. Goldsbrough (1952) determined the sea-level response of a rectangular gulf of constant depth to a specified rise and fall in the sea level at its open end. The horizontal motion of the water was directed along the length of the gulf and the rotation of the Earth was neglected; the boundary condition of zero bottom current was employed. The present paper deals with a similar problem in respect of a rotating shelf sea with a surge input along its oceanic edge. Proudman (1954*b*) obtained solutions of the fundamental differential equations relating to storm surges representing damped Kelvin and damped Poincaré waves in a channel of uniform depth and width; friction and geostrophic effects were taken into account.

Finally, mention is made of a paper by Jeffreys (1923), and one by Hidaka (1953), dealing with the effect of a steady wind on the sea level near the straight coast of a semi-infinite sea. Hidaka considered a steady wind blowing in a finite band within a certain distance from the coast and took horizontal as well as vertical mixing into account.

3. Notation

A continental shelf of uniform depth and width, bounded by a straight infinitely long coast-line, is considered. The following symbols are used:

- h the depth of the shelf
- l its width
- g the acceleration of the Earth's gravity
- Ω the angular speed of the Earth's rotation
- ϕ the latitude
- ρ the density of the water, assumed uniform
- x, y, z Cartesian co-ordinates; axes $Oxyz$ are left-handed: the origin O is a point on the coast in the undisturbed surface of the sea, Ox, Oy lie in this surface with Ox normal to the coastline, and Oz is directed vertically downwards
- ζ the elevation of the water surface
- u, v, w components of the current in the directions of increasing x, y, z respectively
- u_m, v_m the depth-mean values of u, v respectively
- p the pressure at a point in the water
- p_a the atmospheric pressure
- F, G the components in the directions of increasing x, y respectively, of the frictional stress of the water above the depth z on the water below that depth
- F_S, G_S the x, y components of the wind-stress on the sea surface
- F_B, G_B the x, y components of the frictional stress of the water on the bottom
- t the time

The cross-sectional geometry of the shelf is shown in figure 1.

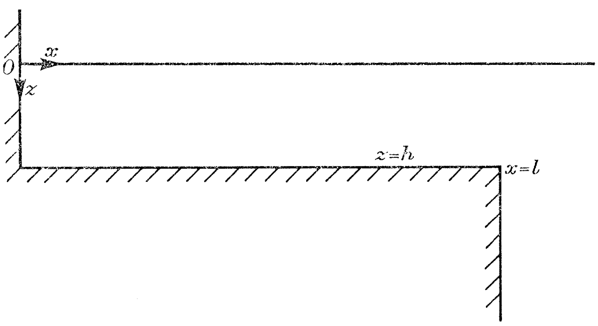

Figure 1. Continental shelf.

4. Basic equations

Equations of continuity and of motion are taken as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$\frac{\partial u}{\partial t} - \gamma v = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x} + \frac{\partial F}{\partial z}\right), \tag{2}$$

$$\frac{\partial v}{\partial t} + \gamma u = -\frac{1}{\rho}\left(\frac{\partial p}{\partial y} + \frac{\partial G}{\partial z}\right), \tag{3}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g, \tag{4}$$

where $\gamma = 2\Omega \sin\phi$, assumed to be a constant. Equations (2), (3) and (4) are deduced from the general equations of motion given by Proudman (1953, p. 96) after ignoring vertical acceleration, the positional terms in the horizontal components of acceleration relative to the Earth, that part of the horizontal component of geostrophic acceleration proportional to w, body forces, and terms involving components of internal stress apart from $-(\partial F/\partial z)/\rho$, $-(\partial G/\partial z)/\rho$.

Integrating (4), satisfying $p = p_a$ when $z = -\zeta$, gives

$$p = p_a + \rho g(z+\zeta). \tag{5}$$

Then (5) in (2), (3) yields

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \gamma v &= -g\frac{\partial \zeta}{\partial x} - \frac{1}{\rho}\frac{\partial p_a}{\partial x} - \frac{1}{\rho}\frac{\partial F}{\partial z}, \\ \frac{\partial v}{\partial t} + \gamma u &= -g\frac{\partial \zeta}{\partial y} - \frac{1}{\rho}\frac{\partial p_a}{\partial y} - \frac{1}{\rho}\frac{\partial G}{\partial z}. \end{aligned}\right\} \tag{6}$$

Integrating equations (1), (6) from the surface $z = -\zeta$ to the bottom $z = h$, dividing by h, and then putting in the conditions

$$\begin{aligned} w(z=-\zeta) &= -\partial\zeta/\partial t, & w(z=h) &= 0, \\ F(z=-\zeta) &= F_S, & F(z=h) &= F_B, \\ G(z=-\zeta) &= G_S, & G(z=h) &= G_B, \end{aligned}$$

we get

$$\left.\begin{aligned} \frac{1}{h}\int_{-\zeta}^{h}\frac{\partial u}{\partial x}\mathrm{d}z + \frac{1}{h}\int_{-\zeta}^{h}\frac{\partial v}{\partial y}\mathrm{d}z + \frac{1}{h}\frac{\partial \zeta}{\partial t} &= 0, \\ \frac{1}{h}\int_{-\zeta}^{h}\frac{\partial u}{\partial t}\mathrm{d}z - \frac{\gamma}{h}\int_{-\zeta}^{h} v\,\mathrm{d}z &= -g\left(1+\frac{\zeta}{h}\right)\frac{\partial \zeta}{\partial x} - \frac{1}{\rho}\left(1+\frac{\zeta}{h}\right)\frac{\partial p_a}{\partial x} + \frac{F_S - F_B}{\rho h}, \\ \frac{1}{h}\int_{-\zeta}^{h}\frac{\partial v}{\partial t}\mathrm{d}z + \frac{\gamma}{h}\int_{-\zeta}^{h} u\,\mathrm{d}z &= -g\left(1+\frac{\zeta}{h}\right)\frac{\partial \zeta}{\partial y} - \frac{1}{\rho}\left(1+\frac{\zeta}{h}\right)\frac{\partial p_a}{\partial y} + \frac{G_S - G_B}{\rho h}, \end{aligned}\right\} \tag{7}$$

or approximately,

$$\left.\begin{aligned} \frac{\partial u_m}{\partial x} + \frac{\partial v_m}{\partial y} + \frac{1}{h}\frac{\partial \zeta}{\partial t} &= 0, \\ \frac{\partial u_m}{\partial t} - \gamma v_m &= -g\frac{\partial \zeta}{\partial x} - \frac{1}{\rho}\frac{\partial p_a}{\partial x} + \frac{F_S - F_B}{\rho h}, \\ \frac{\partial v_m}{\partial t} + \gamma u_m &= -g\frac{\partial \zeta}{\partial y} - \frac{1}{\rho}\frac{\partial p_a}{\partial y} + \frac{G_S - G_B}{\rho h}. \end{aligned}\right\} \tag{8}$$

It is assumed (Proudman 1954*a*) that

$$\left.\begin{aligned} F_B &= 2k\rho h u_m, \\ G_B &= 2k\rho h v_m, \end{aligned}\right\} \tag{9}$$

where k is some constant. Then (8) takes the form

$$\left.\begin{aligned}\frac{\partial u_m}{\partial x}+\frac{\partial v_m}{\partial y}+\frac{1}{h}\frac{\partial \zeta}{\partial t}=0,\\ \frac{\partial u_m}{\partial t}+2ku_m-\gamma v_m=-g\frac{\partial \zeta}{\partial x}-\frac{1}{\rho}\frac{\partial p_a}{\partial x}+\frac{F_S}{\rho h},\\ \frac{\partial v_m}{\partial t}+2kv_m+\gamma u_m=-g\frac{\partial \zeta}{\partial y}-\frac{1}{\rho}\frac{\partial p_a}{\partial y}+\frac{G_S}{\rho h}.\end{aligned}\right\} \tag{10}$$

Supposing that the motion on the shelf is the same for all sections normal to the coast, dependency on y is eliminated and equations (10) reduce to

$$\left.\begin{aligned}\frac{\partial u_m}{\partial x}+\frac{1}{h}\frac{\partial \zeta}{\partial t}=0,\\ \frac{\partial u_m}{\partial t}+2ku_m-\gamma v_m=-g\frac{\partial \zeta}{\partial x}-\frac{1}{\rho}\frac{\partial p_a}{\partial x}+\frac{F_S}{\rho h},\\ \frac{\partial v_m}{\partial t}+2kv_m+\gamma u_m=\frac{G_S}{\rho h}.\end{aligned}\right\} \tag{11}$$

5. Surge input from the ocean

We shall determine the dynamic response of the water on the continental shelf to a prescribed time-variation in the sea level at the oceanic edge, written

$$\zeta=f(t) \quad \text{at} \quad x=l. \tag{12}$$

The variation is taken to represent an oceanic surge incident on the shelf. This assumes that ζ at $x=l$ is completely determined by incoming oceanic disturbances.

It is supposed that wind stresses and horizontal atmospheric pressure gradients over the sea surface are zero, so that equations (11), which we assume are applicable, reduce to

$$\left.\begin{aligned}\frac{\partial u_m}{\partial x}+\frac{1}{h}\frac{\partial \zeta}{\partial t}=0,\\ \frac{\partial u_m}{\partial t}+2ku_m-\gamma v_m=-g\frac{\partial \zeta}{\partial x},\\ \frac{\partial v_m}{\partial t}+2kv_m+\gamma u_m=0.\end{aligned}\right\} \tag{13}$$

Initially the water is considered to be at rest, in equilibrium, and therefore

$$u_m=v_m=\zeta=0 \quad \text{when} \quad t=0. \tag{14}$$

With these conditions, it follows that taking Laplace transforms in (13) yields

$$\left.\begin{aligned}\frac{\mathrm{d}\bar{u}_m}{\mathrm{d}x}+\frac{s}{h}\bar{\zeta}=0,\\ (s+2k)\,\bar{u}_m-\gamma\bar{v}_m=-g\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}x},\\ (s+2k)\,\bar{v}_m+\gamma\bar{u}_m=0,\end{aligned}\right\} \tag{15}$$

where the Laplace transform of a function $R(x, t)$ is defined by

$$\bar{R}(x, s) = \int_0^\infty e^{-st} R(x, t)\, dt.$$

Eliminating $\bar{v}_m$, $\bar{\zeta}$ from equations (15) gives

$$d^2\bar{u}_m/dx^2 = \alpha^2\bar{u}_m, \tag{16}$$

where

$$\alpha^2 = \frac{s[(s+2k)^2+\gamma^2]}{(s+2k)\,gh}. \tag{17}$$

Solving (16) and then using (15) to express $\bar{v}_m$ and $\bar{\zeta}$ in terms of $\bar{u}_m$, we get

$$\left.\begin{aligned} \bar{u}_m &= A\cosh\alpha x + B\sinh\alpha x,\\ \bar{v}_m &= -[\gamma/(s+2k)]\,(A\cosh\alpha x + B\sinh\alpha x),\\ \bar{\zeta} &= -(\alpha h/s)\,(A\sinh\alpha x + B\cosh\alpha x).\end{aligned}\right\} \tag{18}$$

The condition of zero normal flow to be satisfied along the coastline is

$$u_m = 0 \quad \text{at} \quad x = 0. \tag{19}$$

From (19), (12):

$$\left.\begin{aligned} \bar{u}_m &= 0 \quad \text{at} \quad x = 0,\\ \bar{\zeta} &= \bar{f}(s) \quad \text{at} \quad x = l.\end{aligned}\right\} \tag{20}$$

Combining (18) and (20) the constants A, B are determined and $\bar{\zeta}$, $\bar{u}_m$, $\bar{v}_m$ are obtained as follows:

$$\left.\begin{aligned} \bar{\zeta} &= \bar{f}(s)\frac{\cosh\alpha x}{\cosh\alpha l},\\ \bar{u}_m &= -\frac{s\bar{f}(s)}{h\alpha}\frac{\sinh\alpha x}{\cosh\alpha l},\\ \bar{v}_m &= \frac{\gamma s\bar{f}(s)}{h\alpha(s+2k)}\frac{\sinh\alpha x}{\cosh\alpha l}.\end{aligned}\right\} \tag{21}$$

The inversion theorem of the Laplace transformation then gives

$$\zeta = \frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty} \bar{f}(s)\frac{\cosh\alpha x}{\cosh\alpha l}e^{st}\,ds, \tag{22·1}$$

$$u_m = -\frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty} \frac{s\bar{f}(s)}{h\alpha}\frac{\sinh\alpha x}{\cosh\alpha l}e^{st}\,ds, \tag{22·2}$$

and

$$v_m = \frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty} \frac{\gamma s\bar{f}(s)}{h\alpha(s+2k)}\frac{\sinh\alpha x}{\cosh\alpha l}e^{st}\,ds. \tag{22·3}$$

In each integral, γ_1 is real and positive, greater than the real parts of all the singularities of the integrand.

6. Surges generated on the shelf by wind

Consider the motion generated on the continental shelf by a steady uniform wind-stress field created suddenly at time $t = 0$ over the area of sea surface between $x = a$ and $x = l$. Suppose that the wind stress has an onshore component of magnitude P, and a longshore component (in the direction of Oy) of magnitude Q. Here, a, P, Q denote constant values (figure 2). Then

$$\left.\begin{aligned} \partial p_a/\partial x &= 0, \\ F_S = G_S &= 0 \quad (0 < x < a), \\ F_S = -PH(t), \quad G_S &= QH(t) \quad (a < x < l), \end{aligned}\right\} \tag{23}$$

where $H(t)$ denotes Heavyside's unit function. Under these conditions equations (11), assumed again to be applicable, become, for $0 < x < a$,

$$\left.\begin{aligned} \frac{\partial u_m}{\partial x} + \frac{1}{h}\frac{\partial \zeta}{\partial t} &= 0, \\ \frac{\partial u_m}{\partial t} + 2ku_m - \gamma v_m &= -g\frac{\partial \zeta}{\partial x}, \\ \frac{\partial v_m}{\partial t} + 2kv_m + \gamma u_m &= 0; \end{aligned}\right\} \tag{24}$$

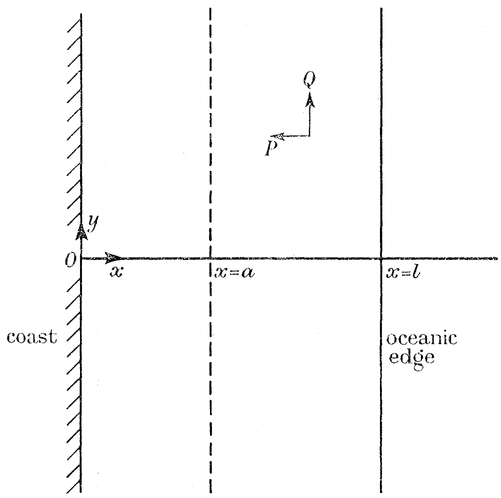

FIGURE 2. Continental shelf, showing an area between $x = a$ and $x = l$ over which the wind stress has an onshore component P and a longshore component Q.

and for $a < x < l$,

$$\left.\begin{aligned} \frac{\partial u_m}{\partial x} + \frac{1}{h}\frac{\partial \zeta}{\partial t} &= 0, \\ \frac{\partial u_m}{\partial t} + 2ku_m - \gamma v_m &= -g\frac{\partial \zeta}{\partial x} - \frac{P}{\rho h}H(t), \\ \frac{\partial v_m}{\partial t} + 2kv_m + \gamma u_m &= \frac{Q}{\rho h}H(t). \end{aligned}\right\} \tag{25}$$

Assuming that the water is at rest initially, the condition given by (14) is again satisfied, and therefore taking Laplace transforms in (24) and (25) yields, for $0 < x < a$,

$$\left.\begin{aligned} \frac{\mathrm{d}\bar{u}_m}{\mathrm{d}x} + \frac{s}{h}\bar{\zeta} &= 0, \\ (s+2k)\,\bar{u}_m - \gamma \bar{v}_m &= -g\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}x}, \\ (s+2k)\,\bar{v}_m + \gamma \bar{u}_m &= 0; \end{aligned}\right\} \tag{26}$$

and for $a < x < l$,

$$\left.\begin{aligned}\frac{d\bar{u}_m}{dx}+\frac{s}{h}\bar{\zeta}&=0,\\(s+2k)\,\bar{u}_m-\gamma\bar{v}_m&=-g\frac{d\bar{\zeta}}{dx}-\frac{P}{\rho hs},\\(s+2k)\,\bar{v}_m+\gamma\bar{u}_m&=\frac{Q}{\rho hs}.\end{aligned}\right\}\qquad(27)$$

The general solution to equations (26) has already been found and is given by (18). Hence, for $0 < x < a$ we may take

$$\left.\begin{aligned}\bar{u}_m&=A_1\cosh\alpha x+B_1\sinh\alpha x,\\\bar{v}_m&=-[\gamma/(s+2k)]\,(A_1\cosh\alpha x+B_1\sinh\alpha x),\\\bar{\zeta}&=-(\alpha h/s)\,(A_1\sinh\alpha x+B_1\cosh\alpha x).\end{aligned}\right\}\qquad(28)$$

Eliminating $\bar{v}_m$, $\bar{\zeta}$ from equations (27) gives

$$\frac{d^2\bar{u}_m}{dx^2}-\alpha^2\bar{u}_m=\frac{P(s+2k)-\gamma Q}{\rho gh^2(s+2k)}.\qquad(29)$$

Solving this equation, and then using (27) to obtain $\bar{v}_m$ and $\bar{\zeta}$, it follows that, for $a < x < l$,

$$\left.\begin{aligned}\bar{u}_m&=A_2\cosh\alpha x+B_2\sinh\alpha x-\frac{\{P(s+2k)-\gamma Q\}}{\rho gh^2\alpha^2(s+2k)},\\\bar{v}_m&=-[\gamma/(s+2k)]\,(A_2\cosh\alpha x+B_2\sinh\alpha x)+\frac{P\gamma(s+2k)-\gamma^2Q}{\rho gh^2\alpha^2(s+2k)^2}+\frac{Q}{\rho hs(s+2k)},\\\bar{\zeta}&=-(\alpha h/s)\,(A_2\sinh\alpha x+B_2\cosh\alpha x).\end{aligned}\right\}\qquad(30)$$

In the absence of surges coming in from the ocean we assume that

$$\zeta=0\quad\text{at}\quad x=l.\qquad(31)$$

From this and the coastal condition (19) it is deduced that

$$\left.\begin{aligned}\bar{\zeta}&=0\quad\text{at}\quad x=l,\\\bar{u}_m&=0\quad\text{at}\quad x=0.\end{aligned}\right\}\qquad(32)$$

Also, continuity of flow and pressure requires the continuity of u_m and ζ, and hence of $\bar{u}_m$ and $\bar{\zeta}$, at $x=a$. These conditions applied to (28) and (30) determine the constants A_1, B_1, A_2, B_2. Subsequently it is found that for $0 < x < a$,

$$\left.\begin{aligned}\bar{u}_m&=-\frac{[P(s+2k)-\gamma Q]}{\rho gh^2\alpha^2\,(s+2k)}\,\frac{\sinh\{\alpha(l-a)\}\sinh\alpha x}{\cosh\alpha l},\\\bar{v}_m&=\frac{\gamma[P(s+2k)-\gamma Q]}{\rho gh^2\alpha^2(s+2k)^2}\,\frac{\sinh\{\alpha(l-a)\}\sinh\alpha x}{\cosh\alpha l},\\\bar{\zeta}&=\frac{[P(s+2k)-\gamma Q]}{\rho gh\alpha s(s+2k)}\,\frac{\sinh\{\alpha(l-a)\}\cosh\alpha x}{\cosh\alpha l},\end{aligned}\right\}\qquad(33)$$

and for $a < x < l$,

$$\left.\begin{aligned}\bar{u}_m&=\frac{P(s+2k)-\gamma Q}{\rho gh^2\alpha^2(s+2k)}\left[\frac{\cosh\alpha a\cosh\{\alpha(l-x)\}}{\cosh\alpha l}-1\right],\\\bar{v}_m&=-\frac{\gamma[P(s+2k)-\gamma Q]}{\rho gh^2\alpha^2(s+2k)^2}\left[\frac{\cosh\alpha a\cosh\{\alpha(l-x)\}}{\cosh\alpha l}-1\right]+\frac{Q}{\rho hs(s+2k)},\\\bar{\zeta}&=\frac{[P(s+2k)-\gamma Q]}{\rho gh\alpha s(s+2k)}\,\frac{\cosh\alpha a\sinh\{\alpha(l-x)\}}{\cosh\alpha l}.\end{aligned}\right\}\qquad(34)$$

Inverting these expressions we get
for $0 < x < a$,

$$u_m = -\frac{1}{2\pi \mathrm{i}} \frac{1}{\rho g h^2} \int_{\gamma_1 - \mathrm{i}\infty}^{\gamma_1 + \mathrm{i}\infty} [P(s+2k) - \gamma Q] \frac{\sinh\{\alpha(l-a)\} \sinh \alpha x}{\alpha^2 (s+2k) \cosh \alpha l} \mathrm{e}^{st} \mathrm{d}s, \tag{35·1}$$

$$v_m = \frac{1}{2\pi \mathrm{i}} \frac{\gamma}{\rho g h^2} \int_{\gamma_1 - \mathrm{i}\infty}^{\gamma_1 + \mathrm{i}\infty} [P(s+2k) - \gamma Q] \frac{\sinh\{\alpha(l-a)\} \sinh \alpha x}{\alpha^2 (s+2k)^2 \cosh \alpha l} \mathrm{e}^{st} \mathrm{d}s, \tag{35·2}$$

$$\zeta = \frac{1}{2\pi \mathrm{i}} \frac{1}{\rho g h} \int_{\gamma_1 - \mathrm{i}\infty}^{\gamma_1 + \mathrm{i}\infty} [P(s+2k) - \gamma Q] \frac{\sinh\{\alpha(l-a)\} \cosh \alpha x}{\alpha s\,(s+2k) \cosh \alpha l} \mathrm{e}^{st} \mathrm{d}s; \tag{35·3}$$

and for $a < x < l$,

$$u_m = \frac{1}{2\pi \mathrm{i}} \frac{1}{\rho g h^2} \int_{\gamma_1 - \mathrm{i}\infty}^{\gamma_1 + \mathrm{i}\infty} \frac{P(s+2k) - \gamma Q}{\alpha^2 (s+2k)} \left[\frac{\cosh \alpha a \cosh\{\alpha(l-x)\}}{\cosh \alpha l} - 1\right] \mathrm{e}^{st} \mathrm{d}s, \tag{35·4}$$

$$v_m = -\frac{1}{2\pi \mathrm{i}} \frac{\gamma}{\rho g h^2} \int_{\gamma_1 - \mathrm{i}\infty}^{\gamma_1 + \mathrm{i}\infty} \frac{P(s+2k) - \gamma Q}{\alpha^2 (s+2k)^2} \left[\frac{\cosh \alpha a \cosh\{\alpha(l-x)\}}{\cosh \alpha l} - 1\right] \mathrm{e}^{st} \mathrm{d}s + \frac{Q}{2k\rho h}(1 - \mathrm{e}^{-2kt}), \tag{35·5}$$

$$\zeta = \frac{1}{2\pi \mathrm{i}} \frac{1}{\rho g h} \int_{\gamma_1 - \mathrm{i}\infty}^{\gamma_1 + \mathrm{i}\infty} [P(s+2k) - \gamma Q] \frac{\cosh \alpha a \sinh\{\alpha\,(l-x)\}}{\alpha s (s+2k) \cosh \alpha l} \mathrm{e}^{st} \mathrm{d}s. \tag{35·6}$$

7. Elevation on the shelf produced by an oceanic surge

In equation (12) take

$$f(t) = H(t)\, \zeta_m \sin \omega t. \tag{36}$$

According to this relation the elevation of the sea surface at the oceanic edge of the shelf rises from zero at $t = 0$ and subsequently executes simple harmonic oscillations of amplitude ζ_m and period $2\pi/\omega$ about its zero value. The resulting elevation on the shelf is, from (22·1), given by

$$\frac{\zeta}{\zeta_m} = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1 - \mathrm{i}\infty}^{\gamma_1 + \mathrm{i}\infty} \frac{\omega}{s^2 + \omega^2} \frac{\cosh \alpha x}{\cosh \alpha l} \mathrm{e}^{st} \mathrm{d}s. \tag{37}$$

The problem now is to evaluate this integral using the calculus of residues. The integrand, a single-valued function of s, has singularities at $s = \pm \mathrm{i}\omega$ and at the roots of

$$\cosh \alpha l = 0.$$

Solving this equation gives

$$\alpha l = \pm \tfrac{1}{2} \mathrm{i} (2n-1)\, \pi \quad (n = 1, 2, 3, \ldots), \tag{38}$$

from which, using (17), we get

$$s\left(s + 2k + \frac{\gamma^2}{s+2k}\right) = -gh(2n-1)^2 \frac{\pi^2}{4l^2}. \tag{39}$$

Equation (39) is a cubic in s. When

$$\tfrac{16}{3} k^2 < 4\gamma^2 + gh(2n-1)^2 \pi^2/l^2 \tag{40}$$

its roots are of the form

$$-\lambda_n, \quad -\mu_n \pm \mathrm{i}\nu_n, \tag{41}$$

where λ_n, μ_n, ν_n are real and positive such that

$$0 < \lambda_n < 2k, \tag{42}$$

$$\mu_n = 2k - \tfrac{1}{2}\lambda_n, \tag{43}$$

$$\nu_n = \tfrac{1}{2}\{3\lambda_n^2 - 8k\lambda_n + 4\gamma^2 + gh(2n-1)^2 \pi^2/l^2\}^{\frac{1}{2}}. \tag{44}$$

It is expected that the inequality (40) will be satisfied for all n over a wide range of natural conditions, for it certainly holds when $\frac{16}{3}k^2 < 4\gamma^2$, i.e. when $\sin\phi > k/\Omega\surd 3$, which, on taking $k = 0{\cdot}045/\mathrm{h}$ (following Lauwerier 1957 *a*) and $\Omega = 0{\cdot}2625/\mathrm{h}$, gives $\phi > 5°\,41'$. We shall therefore assume that the roots of (39) are of the form given by (41).

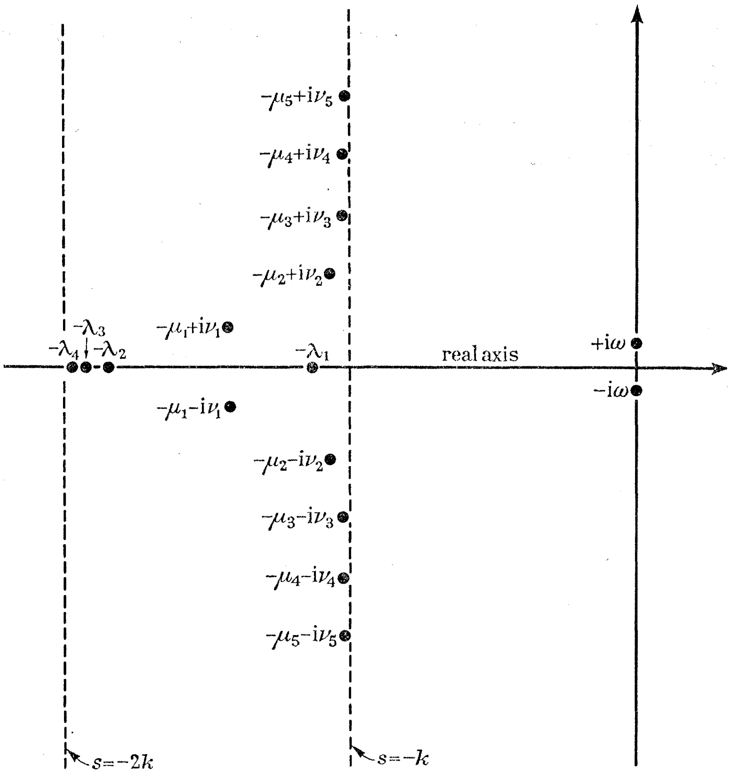

FIGURE 3. Complex s-plane showing poles of the integrand L.

It follows that the integrand in (37) has simple poles at

$$s = \pm i\omega, \quad -\lambda_n, \quad -\mu_n \pm i\nu_n \quad (n = 1, 2, 3, \ldots).$$

From (39),
$$-\lambda_n = -2k + \frac{8kl^2\gamma^2}{gh(2n-1)^2\pi^2} + O\left(\frac{1}{(2n-1)^4}\right), \tag{45}$$

and therefore as $n \to \infty$, $-\lambda_n \to -2k$. The sequence $-\lambda_1, -\lambda_2, -\lambda_3, \ldots$ is monotonic decreasing and the limit point $-2k$ is a non-isolated essential singularity of the integrand (Phillips 1945, p. 101). From (42), (43), (44): $-2k < -\mu_n < -k$ for all n, $-\mu_n \to -k$ and $\nu_n \to \infty$ as $n \to \infty$ (figure 3).

For convenience in the work which follows equation (37) is written

$$\frac{\zeta}{\zeta_m} = \frac{1}{2\pi i}\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} L\,ds, \tag{46}$$

where
$$L = \frac{\omega}{s^2+\omega^2}\,\frac{\cosh\alpha x}{\cosh\alpha l}\,e^{st}. \tag{47}$$

We consider the contour integral

$$J = \frac{1}{2\pi i}\int_\Gamma L\,ds, \tag{48}$$

where Γ is a closed contour in the complex s-plane (figure 4) consisting of:

(i) the straight line joining $\gamma_1 - i\surd(R^2-\gamma_1^2)$ to $\gamma_1 + i\surd(R^2-\gamma_1^2)$ denoted by Γ_1; R is greater than γ_1, $2k$, and ω;

(ii) that part of the circle $s = R e^{i\theta}$ lying to the left of this line, denoted by Γ_2;

(iii) a circle of radius r with its centre at $s = -2k$, denoted by Γ_3; $r < R-2k$ and $r < 2k - \mu_1$;

(iv) the upper and lower edges of a narrow slit extending along the real axis between $s = -2k-r$ and $s = -R$.

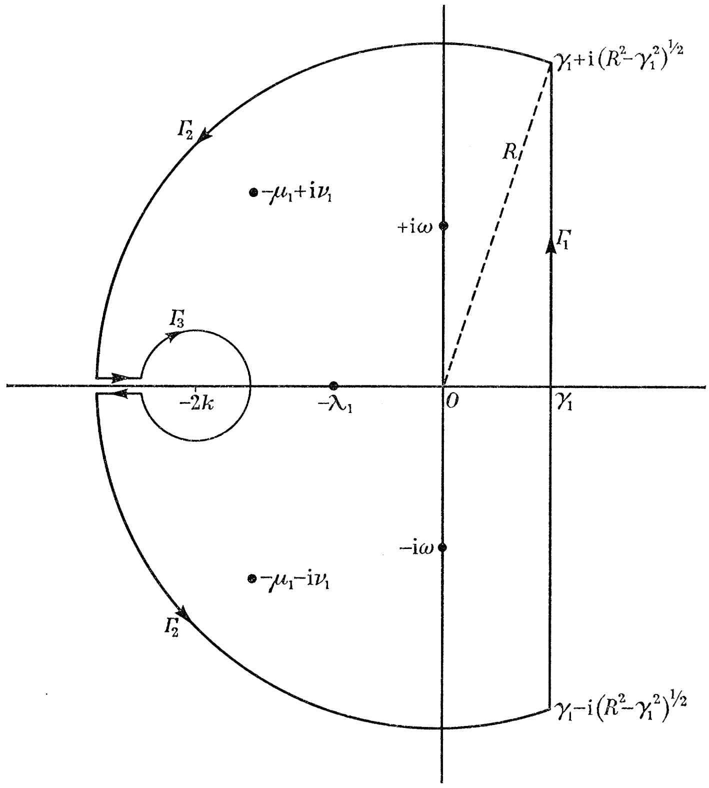

FIGURE 4. Diagram of the complex s-plane showing the closed contour Γ.

The contribution to J coming from the integrals along the edges of the slit is zero and therefore, by Cauchy's residue theorem,

$$\left.\begin{aligned} J &= \frac{1}{2\pi i}\left[\int_{\Gamma_1} L\,ds + \int_{\Gamma_2} L\,ds + \int_{\Gamma_3} L\,ds\right] \\ &= \text{sum of the residues at the poles of } L \text{ which lie within } \Gamma. \end{aligned}\right\} \tag{49}$$

Consider now the non-zero roots of

$$|\cosh \alpha l| = 1.$$

These satisfy $$\alpha l = \pm i n\pi \quad (n = 1, 2, 3, \ldots),$$

i.e. from (17) $$s\{s + 2k + \gamma^2/(s+2k)\} = -ghn^2\pi^2/l^2.$$

It is assumed that (40) is satisfied and therefore this equation has roots, similar to those of (39), of the form $$-\lambda_n', \quad -\mu_n' \pm i\nu_n' \quad (n = 1, 2, 3, \ldots),$$

where λ'_n, μ'_n, ν'_n are real and positive such that

$$0 < \lambda'_n < 2k \quad (\lambda'_n \to 2k \quad \text{as} \quad n \to \infty),$$

$$\mu'_n = 2k - \tfrac{1}{2}\lambda'_n,$$

$$\nu'_n = \tfrac{1}{2}\{3\lambda'^2_n - 8k\lambda'_n + 4\gamma^2 + 4ghn^2\pi^2/l^2\}^{\frac{1}{2}}.$$

We take
$$r = 2k - \lambda'_n \tag{50}$$

$$R = (\mu'^2_n + \nu'^2_n)^{\frac{1}{2}}. \tag{51}$$

Then, as $n \to \infty$, $r \to 0$ and $R \to \infty$; n may be chosen large enough to satisfy the conditions already imposed on the magnitudes of r, R. From (50), Γ_3 passes through $s = -\lambda'_n$ and, using an expansion for $-\lambda'_n$ similar to that for $-\lambda_n$ given by (45), we get

$$r = 2kl^2\gamma^2/ghn^2\pi^2 + O(n^{-4}).$$

If r_N denotes the radius of the circle concentric with Γ_3 through $-\lambda_N$,

$$r_N = 2k - \lambda_N = 2kl^2\gamma^2/gh(N-\tfrac{1}{2})^2\pi^2 + O\{(N-\tfrac{1}{2})^{-4}\}.$$

It follows that, for n large enough,

$$r_N > r \quad (N = 1, 2, 3, \ldots, n);$$

$$r_N < r \quad (N = n+1, n+2, n+3, \ldots).$$

From (51), Γ_2 passes through $s = -\mu'_n \pm \mathrm{i}\nu'_n$ and

$$R = [(2k-\lambda'_n)^2 + \gamma^2 + ghn^2\pi^2/l^2]^{\frac{1}{2}}$$

$$= n\pi\surd(gh)/l + O(1/n).$$

If R_N denotes the radius of the circle concentric with Γ_2 through $-\mu_N \pm \mathrm{i}\nu_N$,

$$R_N = (\mu^2_N + \nu^2_N)^{\frac{1}{2}}$$

$$= [(2k-\lambda_N)^2 + \gamma^2 + gh(N-\tfrac{1}{2})^2\pi^2/l^2]^{\frac{1}{2}}$$

$$= (N-\tfrac{1}{2})\,\pi\surd(gh)/l + O\{(N-\tfrac{1}{2})^{-1}\}.$$

It follows that, for sufficiently large n,

$$R_N < R, \quad N = 1,\ 2,\ 3, \ldots, n;$$

$$R_N > R, \quad N = n+1, n+2, n+3, \ldots.$$

Hence, with r, R given by (50), (51), for large n the poles

$$-\lambda_1, -\lambda_2, \ldots, -\lambda_n; \quad -\mu_1 \pm \mathrm{i}\nu_1, -\mu_2 \pm \mathrm{i}\nu_2, \ldots, -\mu_n \pm \mathrm{i}\nu_n$$

lie within Γ, and the poles

$$-\lambda_{n+1}, -\lambda_{n+2}, \ldots; \quad -\mu_{n+1} \pm \mathrm{i}\nu_{n+1}, -\mu_{n+2} \pm \mathrm{i}\nu_{n+2}, \ldots$$

outside it.

We have, therefore, that as $n \to \infty$ through sufficiently large values, $R \to \infty$ and $r \to 0$, and Γ never passes through a pole of the integrand L. In proceeding to the limit under these conditions,

$$\left|\frac{\cosh \alpha x}{\cosh \alpha l}\right|$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

is bounded on Γ_2 and Γ_3, and as a consequence it may be shown that

$$\int_{\Gamma_2} L\,\mathrm{d}s \to 0, \quad \int_{\Gamma_3} L\,\mathrm{d}s \to 0.$$

Also,
$$\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_1} L\,\mathrm{d}s \to \frac{1}{2\pi \mathrm{i}}\int_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty} L\,\mathrm{d}s = \frac{\zeta}{\zeta_m}.$$

Hence, from (49)

$$\zeta/\zeta_m = \text{sum of the residues at all the poles of } L. \tag{52}$$

The residues are determined as follows:

(Residue at $s = \mathrm{i}\omega$) + (residue at $s = -\mathrm{i}\omega$)

$$= \frac{1}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}\omega t}\left(\frac{\cosh \alpha x}{\cosh \alpha l}\right)_{s=\mathrm{i}\omega} - \mathrm{e}^{-\mathrm{i}\omega t}\left(\frac{\cosh \alpha x}{\cosh \alpha l}\right)_{s=-\mathrm{i}\omega}\right]$$

$$= 2\frac{\left\{\begin{array}{l}(\cosh \xi x \cosh \xi l \cos \eta x \cos \eta l + \sinh \xi x \sinh \xi l \sin \eta x \sin \eta l) \sin \omega t \\ \qquad + (\sinh \xi x \cosh \xi l \sin \eta x \cos \eta l - \cosh \xi x \sinh \xi l \cos \eta x \sin \eta l) \cos \omega t\end{array}\right\}}{\cosh 2\xi l + \cos 2\eta l},$$

where
$$\xi = \sigma \cos \tfrac{1}{2}\psi, \quad \eta = \sigma \sin \tfrac{1}{2}\psi, \tag{53}$$

in which
$$\left.\begin{array}{l}\sigma = \left[\dfrac{\omega^2\{(\gamma-\omega)^2+4k^2\}\{(\gamma+\omega)^2+4k^2\}}{g^2h^2(\omega^2+4k^2)}\right]^{\frac{1}{4}}, \\ \psi = \tan^{-1}\left[\dfrac{2k(\gamma^2+4k^2+\omega^2)}{\omega(\gamma^2-4k^2-\omega^2)}\right].\end{array}\right\} \tag{54}$$

It should be noted that when inverse tangents are given in the form

$$\tan^{-1}(\Upsilon_1/\Upsilon_2) = \Upsilon, \text{ say,}$$

$\sin \Upsilon$ has the sign of Υ_1, and $\cos \Upsilon$ the sign of Υ_2.

$$(\text{Residue at } s = -\lambda_n) = \frac{\omega}{\omega^2+\lambda_n^2}\left[\frac{\cosh \alpha x}{\mathrm{d}(\cosh \alpha l)/\mathrm{d}s}\right]_{s=-\lambda_n} \mathrm{e}^{-\lambda_n t}$$

$$= (-1)^{n+1}\frac{\pi g h \omega (2n-1)(2k-\lambda_n)^2}{2l^2(\omega^2+\lambda_n^2)[(k-\lambda_n)(2k-\lambda_n)^2+k\gamma^2]}\cos\left\{(2n-1)\frac{\pi x}{2l}\right\}\mathrm{e}^{-\lambda_n t}.$$

(Residue at $s = -\mu_n + \mathrm{i}\nu_n$) + (residue at $s = -\mu_n - \mathrm{i}\nu_n$)

$$= 2 \times (\text{real part of residue at } s = -\mu_n + \mathrm{i}\nu_n)$$

$$= \mathscr{R}\left\{\frac{2\omega}{\omega^2+(-\mu_n+\mathrm{i}\nu_n)^2}\left[\frac{\cosh \alpha x}{\mathrm{d}(\cosh \alpha l)/\mathrm{d}s}\right]_{s=-\mu_n+\mathrm{i}\nu_n}\mathrm{e}^{(-\mu_n+\mathrm{i}\nu_n)t}\right\}$$

$$= (-1)^{n+1}\frac{\pi g h \omega (2n-1)}{l^2 D_n E_n}\cos\left\{(2n-1)\frac{\pi x}{2l}\right\}\cos(\nu_n t - \Theta_n - \chi_n)\,\mathrm{e}^{-\mu_n t},$$

where
$$\left.\begin{array}{l}D_n = \{(\omega^2+\mu_n^2-\nu_n^2)^2+4\mu_n^2\nu_n^2\}^{\frac{1}{2}}, \\ E_n = \{[k-\mu_n+(k\gamma^2/d_n^2)\cos 2\theta_n]^2+[\nu_n-(k\gamma^2/d_n^2)\sin 2\theta_n]^2\}^{\frac{1}{2}}, \\ \Theta_n = \tan^{-1}\left[\dfrac{-2\mu_n\nu_n}{\omega^2+\mu_n^2-\nu_n^2}\right], \\ \chi_n = \tan^{-1}\left[\dfrac{\nu_n-(k\gamma^2/d_n^2)\sin 2\theta_n}{k-\mu_n+(k\gamma^2/d_n^2)\cos 2\theta_n}\right],\end{array}\right\} \tag{55}$$

in which
$$\left.\begin{aligned} d_n &= \{(2k-\mu_n)^2+\nu_n^2\}^{\frac{1}{2}}, \\ \theta_n &= \tan^{-1}[\nu_n/(2k-\mu_n)]. \end{aligned}\right\} \tag{56}$$

Summing the residues we get, from (52),

$$\frac{\zeta}{\zeta_m} = \left\{\begin{aligned} &\frac{2\left\{\begin{aligned}&(\cosh\xi x\cosh\xi l\cos\eta x\cos\eta l+\sinh\xi x\sinh\xi l\sin\eta x\sin\eta l)\sin\omega t\\ &+(\sinh\xi x\cosh\xi l\sin\eta x\cos\eta l-\cosh\xi x\sinh\xi l\cos\eta x\sin\eta l)\cos\omega t\end{aligned}\right\}}{\cosh 2\xi l+\cos 2\eta l} \\ &+\frac{\pi g h\omega}{2l^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n-1)(2k-\lambda_n)^2}{(\omega^2+\lambda_n^2)[(k-\lambda_n)(2k-\lambda_n)^2+k\gamma^2]}\cos\left\{(2n-1)\frac{\pi x}{2l}\right\}e^{-\lambda_n t} \\ &+\frac{\pi g h\omega}{l^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n-1)}{D_n E_n}\cos\left\{(2n-1)\frac{\pi x}{2l}\right\}\cos(\nu_n t-\Theta_n-\chi_n)\,e^{-\mu_n t}. \end{aligned}\right. \tag{57}$$

This gives the elevation $\zeta = \zeta(x,t)$ due to motion generated from a state of rest at $t=0$ by an oscillation $\zeta = \zeta_m \sin\omega t$ applied at $x = l$ for $t \geqslant 0$. It follows that for an oceanic surge incident on the shelf, represented by a simple rise and fall of the water level at $x = l$ given by the half wave
$$\left.\begin{aligned} \zeta &= \zeta_m\sin\omega t \quad && (0\leqslant t\leqslant \pi/\omega) \\ &= 0 && (t\geqslant \pi/\omega), \end{aligned}\right\}$$
the elevation on the shelf is
$$\left.\begin{aligned} \zeta &= \zeta(x,t) && (0\leqslant t\leqslant \pi/\omega) \\ &= \zeta(x,t)+\zeta(x,t-\pi/\omega) \quad && (t\geqslant \pi/\omega). \end{aligned}\right\}$$

It is easily seen that ζ/ζ_m given by (57) may be rewritten in terms of the following non-dimensional parameters:

$$\left.\begin{aligned} &\hat{x} = x/l, \quad \hat{\xi} = \xi l, \quad \hat{\eta} = \eta l, \\ &\hat{t} = t(\surd(gh)/l), \\ &\hat{k} = k(l/\surd(gh)), \quad \hat{\gamma} = \gamma(l/\surd(gh)), \quad \hat{\omega} = \omega(l/\surd(gh)), \\ &\hat{\lambda}_n = \lambda_n(l/\surd(gh)), \quad \hat{\mu}_n = \mu_n(l/\surd(gh)), \quad \hat{\nu}_n = \nu_n(l/\surd(gh)). \end{aligned}\right\} \tag{58}$$

It follows from (39) that $-\hat{\lambda}_n$, $-\hat{\mu}_n \pm \mathrm{i}\hat{\nu}_n$ are the roots of

$$\hat{s}(\hat{s}+2\hat{k}+\hat{\gamma}^2/(\hat{s}+2\hat{k})) = -\tfrac{1}{4}(2n-1)^2\pi^2, \tag{59}$$

where $\hat{s}$ is the variable, related to s by

$$\hat{s} = s(l/\surd(gh)). \tag{60}$$

8. Elevation due to a stationary wind field suddenly created over part of the shelf

The integrals (35·3), (35·6) are now evaluated. The elevation of the sea surface on the shelf produced by a steady uniform wind stress field, with onshore component P and longshore component Q, created suddenly at time $t=0$ over the area of the shelf between $x = a$ and $x = l$, is thus determined.

From (35·3), for $0 < x < a$ we have

$$\zeta = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty} M\,\mathrm{d}s, \tag{61}$$

where $$M = \frac{1}{\rho g h}[P(s+2k)-\gamma Q]\frac{\sinh\{\alpha(l-a)\}\cosh\alpha x}{\alpha s(s+2k)\cosh\alpha l}\,\mathrm{e}^{st}. \tag{62}$$

Here, M is a single-valued function of s with simple poles at

$$s = 0, \quad -\lambda_n, \quad -\mu_n \pm \mathrm{i}\nu_n \quad (n = 1, 2, 3, \ldots)$$

and a non-isolated essential singularity at $s = -2k$. Following again the procedure described in §7 we consider the integral of M to be taken round the closed contour Γ in the complex s-plane (figure 4) with r given by (50), and R by (51). It may be shown that as $n \to \infty$ in (50), (51), so that $r \to 0$ and $R \to \infty$, then

$$\int_{\Gamma_2} M\mathrm{d}s \to 0, \quad \int_{\Gamma_3} M\,\mathrm{d}s \to 0,$$

and also $$\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_1} M\mathrm{d}s \to \frac{1}{2\pi\mathrm{i}}\int_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty} M\,\mathrm{d}s = \zeta.$$

Hence, using Cauchy's residue theorem, we deduce that

$$\zeta = \text{sum of the residues at all the poles of } M. \tag{63}$$

The residue at $s = 0$ is $$\left(P-\frac{\gamma Q}{2k}\right)\frac{l-a}{\rho g h}$$

and at $s = -\lambda_n$ is

$$\left\{\frac{1}{\rho g h}[P(s+2k)-\gamma Q]\frac{\sinh\{\alpha(l-a)\}\cosh\alpha x}{\alpha s(s+2k)}\,\frac{\mathrm{e}^{st}}{\mathrm{d}(\cosh\alpha l)/\mathrm{d}s}\right\}_{s=-\lambda_n}$$
$$= (-1)^n\frac{(2k-\lambda_n)\,[P(2k-\lambda_n)-\gamma Q]}{\rho l\lambda_n\,[(k-\lambda_n)\,(2k-\lambda_n)^2+k\gamma^2)]}\sin\left\{\frac{(2n-1)\,\pi\,(l-a)}{2l}\right\}\cos\left\{\frac{(2n-1)\,\pi x}{2l}\right\}\mathrm{e}^{-\lambda_n t}.$$

Also we have:

$$(\text{residue at } s = -\mu_n+\mathrm{i}\nu_n)+(\text{residue at } s = -\mu_n-\mathrm{i}\nu_n)$$
$$= 2\times(\text{real part of residue at } s = -\mu_n+\mathrm{i}\nu_n)$$
$$= 2\mathscr{R}\left\{\left[\frac{1}{\rho g h}[P(s+2k)-\gamma Q]\frac{\sinh\{\alpha(l-a)\}\cosh\alpha x}{\alpha s(s+2k)}\,\frac{\mathrm{e}^{st}}{\mathrm{d}(\cosh\alpha l)/\mathrm{d}s}\right]_{s=-\mu_n+\mathrm{i}\nu_n}\right\}$$
$$= (-1)^n\frac{2I_n}{\rho l E_n}\sin\left\{\frac{(2n-1)\,\pi\,(l-a)}{2l}\right\}\cos\left\{\frac{(2n-1)\,\pi x}{2l}\right\}\cos\,(\nu_n t-\chi_n+\phi_n)\,\mathrm{e}^{-\mu_n t},$$

where $$\left.\begin{aligned} I_n &= \left[\frac{P^2-2PQ\gamma(2k-\mu_n)/d_n^2+\gamma^2Q^2/d_n^2}{\mu_n^2+\nu_n^2}\right]^{\frac{1}{2}}, \\ \phi_n &= \tan^{-1}\left[\frac{(\gamma Q/d_n^2)\,\nu_n}{P-(\gamma Q/d_n^2)\,(2k-\mu_n)}\right]+\tan^{-1}\left(\frac{\nu_n}{\mu_n}\right). \end{aligned}\right\} \tag{64}$$

Hence, from (63), for $0 < x < a$,

$$\zeta = \left(P-\frac{\gamma Q}{2k}\right)\frac{l-a}{\rho g h}$$
$$+\sum_{n=1}^{\infty}(-1)^n\frac{(2k-\lambda_n)\,[P(2k-\lambda_n)-\gamma Q]}{\rho l\lambda_n[(k-\lambda_n)\,(2k-\lambda_n)^2+k\gamma^2]}\sin\left\{\frac{(2n-1)\,\pi\,(l-a)}{2l}\right\}\cos\left\{\frac{(2n-1)\,\pi x}{2l}\right\}\mathrm{e}^{-\lambda_n t}$$
$$+\sum_{n=1}^{\infty}(-1)^n\frac{2I_n}{\rho l E_n}\sin\left\{\frac{(2n-1)\,\pi(l-a)}{2l}\right\}\cos\left\{\frac{(2n-1)\,\pi x}{2l}\right\}\cos\,(\nu_n t-\chi_n+\phi_n)\,\mathrm{e}^{-\mu_n t}. \tag{65}$$

Similarly, evaluating the integral (35·6), we get, for $a < x < l$,

$$\zeta = \left(P - \frac{\gamma Q}{2k}\right)\frac{l-x}{\rho g h}$$
$$+\sum_{n=1}^{\infty}(-1)^n \frac{(2k-\lambda_n)\,[P(2k-\lambda_n)-\gamma Q]}{\rho l\lambda_n[(k-\lambda_n)\,(2k-\lambda_n)^2+k\gamma^2]}\sin\left\{\frac{(2n-1)\,\pi(l-x)}{2l}\right\}\cos\left\{\frac{(2n-1)\pi\,a}{2l}\right\}\mathrm{e}^{-\lambda_n t}$$
$$+\sum_{n=1}^{\infty}(-1)^n\frac{2I_n}{\rho l E_n}\sin\left\{\frac{(2n-1)\,\pi(l-x)}{2l}\right\}\cos\left\{\frac{(2n-1)\,\pi a}{2l}\right\}\cos\,(\nu_n t-\chi_n+\phi_n)\,\mathrm{e}^{-\mu_n t}. \qquad (66)$$

Equations (65), (66) give the elevation of the sea surface over the entire area of the continental shelf at any time following the sudden creation of the wind field between $x = a$ and $x = l$.

9. Coastal elevation due to a wind field with a moving boundary

Suppose that the boundary, $x = a$, of the wind stress field considered in § 8 moves with a velocity V towards the coast, where V is a constant or some function of the time t. Then a is a function of t such that

$$V = -(\mathrm{d}a/\mathrm{d}t). \qquad (67)$$

As the boundary moves towards the coast, the area between $x = a$ and $x = l$ covered by the wind field increases. We consider the case when the boundary traverses the entire width of the shelf in a time T, moving from the oceanic edge $x = l$ at $t = 0$ to the coast-line $x = 0$ at $t = T$. Accordingly, no wind field exists over the shelf at $t = 0$, but a wind area subsequently spreads shorewards from the ocean and covers the shelf when $t \geqslant T$. The variation in sea level at the coast resulting from these changes is now determined.

Let ζ given by (65), evaluated at $x = 0$, be denoted by ζ_0. Then,

$$\zeta_0 = \left(P - \frac{\gamma Q}{2k}\right)\frac{l-a}{\rho g h}$$
$$+\sum_{n=1}^{\infty}(-1)^n\frac{(2k-\lambda_n)\,[P(2k-\lambda_n)-\gamma Q]}{\rho l\lambda_n\,[(k-\lambda_n)\,(2k-\lambda_n)^2+k\gamma^2]}\sin\left\{\frac{(2n-1)\,\pi(l-a)}{2l}\right\}\mathrm{e}^{-\lambda_n t}$$
$$+\sum_{n=1}^{\infty}(-1)^n\frac{2I_n}{\rho l E_n}\sin\left\{\frac{(2n-1)\,\pi(l-a)}{2l}\right\}\cos\,(\nu_n t-\chi_n+\phi_n)\,\mathrm{e}^{-\mu_n t}. \qquad (68)$$

$\zeta_0(t, a)$ thus given is the coastal elevation at time t due to a stress field with components $F_S = -P$, $G_S = Q$ created at $t = 0$ between $x = a$ and $x = l$. Hence

$$\left.\begin{aligned}\zeta_0(t-\tau, a+\mathrm{d}a)-\zeta_0(t-\tau,a)&\\ &=\frac{\partial\zeta_0}{\partial a}(t-\tau,a)\,\mathrm{d}a\\ &=\frac{\partial\zeta_0}{\partial a}(t-\tau,a)\,\frac{\mathrm{d}a}{\mathrm{d}\tau}\mathrm{d}\tau\\ &=-V\frac{\partial\zeta_0}{\partial a}(t-\tau,a)\,\mathrm{d}\tau\end{aligned}\right\}\quad t\geqslant\tau$$

is the coastal elevation due to a field with components of opposite sign created at $t = \tau$ between $x = a$ and $x = a+\mathrm{d}a$, a and V now being regarded as functions of τ. Integrating from $\tau = 0$ to $\tau = t$, it follows that

$$\left.\begin{aligned}\zeta &= -\int_0^t V(\tau)\,\frac{\partial\zeta_0}{\partial a}(t-\tau,a(\tau))\,\mathrm{d}\tau \quad (0\leqslant t\leqslant T)\\ &= -\int_0^T V(\tau)\,\frac{\partial\zeta_0}{\partial a}(t-\tau,a(\tau))\,\mathrm{d}\tau \quad (t\geqslant T)\end{aligned}\right\} \qquad (69)$$

gives the variation in sea level at the coast due to the wind field with the moving boundary defined above. In the case when V is a constant, we have

$$V = l/T, \quad a = l - V\tau \tag{70}$$

and (68) in (69) yields, for $0 \leqslant t \leqslant T$,

$$\begin{aligned}\zeta = {} & \frac{V}{\rho g h}\left(P - \frac{\gamma Q}{2k}\right)t \\ & + \sum_{n=1}^{\infty}(-1)^n \frac{(2k-\lambda_n)\,[P(2k-\lambda_n)-\gamma Q]}{\rho l \lambda_n\,[(k-\lambda_n)\,(2k-\lambda_n)^2 + k\gamma^2]}\,\frac{(2n-1)\,\pi V}{2l}\,\mathrm{e}^{-\lambda_n t}\int_0^t \mathrm{e}^{\lambda_n \tau}\cos\left\{\frac{(2n-1)\,\pi V\tau}{2l}\right\}\mathrm{d}\tau \\ & + \sum_{n=1}^{\infty}(-1)^n \frac{2I_n}{\rho l E_n}\,\frac{(2n-1)\,\pi V}{2l}\,\mathrm{e}^{-\mu_n t}\int_0^t \mathrm{e}^{\mu_n \tau}\cos\left(\nu_n t - \nu_n \tau - \chi_n + \phi_n\right)\cos\left\{\frac{(2n-1)\,\pi V\tau}{2l}\right\}\mathrm{d}\tau\end{aligned}$$

and for $t \geqslant T$,

$$\begin{aligned}\zeta = {} & \frac{V}{\rho g h}\left(P - \frac{\gamma Q}{2k}\right)T \\ & + \sum_{n=1}^{\infty}(-1)^n \frac{(2k-\lambda_n)\,[P(2k-\lambda_n)-\gamma Q]}{\rho l \lambda_n\,[(k-\lambda_n)\,(2k-\lambda_n)^2 + k\gamma^2]}\,\frac{(2n-1)\,\pi V}{2l}\,\mathrm{e}^{-\lambda_n t}\int_0^T \mathrm{e}^{\lambda_n \tau}\cos\left\{\frac{(2n-1)\,\pi V\tau}{2l}\right\}\mathrm{d}\tau \\ & + \sum_{n=1}^{\infty}(-1)^n \frac{2I_n}{\rho l E_n}\,\frac{(2n-1)\,\pi V}{2l}\,\mathrm{e}^{-\mu_n t}\int_0^T \mathrm{e}^{\mu_n \tau}\cos\left(\nu_n t - \nu_n \tau - \chi_n + \phi_n\right)\cos\left\{\frac{(2n-1)\,\pi V\tau}{2l}\right\}\mathrm{d}\tau.\end{aligned}$$

After evaluating the integrals in these expressions we get, for $0 \leqslant t \leqslant T$,

$$\begin{aligned}\zeta = {} & \left(P - \frac{\gamma Q}{2k}\right)\frac{Vt}{\rho g h} \\ & + \sum_{n=1}^{\infty}(-1)^n \frac{(2k-\lambda_n)\,[P(2k-\lambda_n)-\gamma Q]}{\rho l \lambda_n\,[(k-\lambda_n)\,(2k-\lambda_n)^2 + k\gamma^2]\,(1+\beta_n^2)} \\ & \qquad \times\left[\sin\left\{\frac{(2n-1)\,\pi t}{2T}\right\} + \beta_n \cos\left\{\frac{(2n-1)\,\pi t}{2T}\right\} - \beta_n \mathrm{e}^{-\lambda_n t}\right] \\ & + \sum_{n=1}^{\infty}(-1)^n \frac{2I_n}{\rho l E_n}\left[J_n \cos\Lambda_n \sin\left\{\frac{(2n-1)\,\pi t}{2T}\right\}\right. \\ & \qquad \left. + K_n \cos\Psi_n \cos\left\{\frac{(2n-1)\,\pi t}{2T}\right\} - K_n \mathrm{e}^{-\mu_n t}\cos\left(\nu_n t + \Psi_n\right)\right];\end{aligned} \tag{71·1}$$

and for $t \geqslant T$,

$$\begin{aligned}\zeta = {} & \left(P - \frac{\gamma Q}{2k}\right)\frac{l}{\rho g h} \\ & + \sum_{n=1}^{\infty}(-1)^n \frac{(2k-\lambda_n)\,[P(2k-\lambda_n)-\gamma Q]}{\rho l \lambda_n\,[(k-\lambda_n)\,(2k-\lambda_n)^2 + k\gamma^2]\,(1+\beta_n^2)}\,[(-1)^{n+1}\mathrm{e}^{-\lambda_n(t-T)} - \beta_n \mathrm{e}^{-\lambda_n t}] \\ & + \sum_{n=1}^{\infty}(-1)^n \frac{2I_n}{\rho l E_n}\,[(-1)^{n+1}J_n \mathrm{e}^{-\mu_n(t-T)}\cos\{\nu_n(t-T)+\Lambda_n\} - K_n \mathrm{e}^{-\mu_n t}\cos(\nu_n t + \Psi_n)],\end{aligned} \tag{71·2}$$

where

$$\left.\begin{aligned} & \beta_n = \frac{2T\lambda_n}{(2n-1)\,\pi}, \quad \delta_n = \frac{2T\mu_n}{(2n-1)\,\pi}, \quad \epsilon_n = \frac{2T\nu_n}{(2n-1)\,\pi}, \\ & J_n = [(1+\delta_n^2-\epsilon_n^2)^2 + 4\delta_n^2\epsilon_n^2]^{-\frac{1}{2}}, \\ & K_n = J_n\,(\delta_n^2+\epsilon_n^2)^{\frac{1}{2}}, \\ & \Delta_n = \tan^{-1}\left(\frac{2\delta_n\epsilon_n}{1+\delta_n^2-\epsilon_n^2}\right), \\ & \Lambda_n = -\chi_n + \phi_n + \Delta_n, \\ & \Psi_n = \Lambda_n - \tan^{-1}\left(\frac{\nu_n}{\mu_n}\right). \end{aligned}\right\} \tag{72}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Equations (71·1), (71·2) give ζ at the coast ($x = 0$) due to a uniform wind stress field with onshore component P and longshore component Q, which extends steadily across the shelf from the ocean behind a straight boundary moving with a uniform velocity V normal to the coast. By definition, Q is directed along the positive y-axis which points to the right of an observer looking towards the coast.

10. Application of the theory

The theory developed in the preceding sections is now used to investigate the generation and propagation of storm surges on the continental shelf lying to the south of Ireland. For theoretical purposes a rectangular area of the shelf sea, $ABCD$, is considered (figure 5). Then, in the mathematical model, the coast line $x = 0$ is assumed to lie along BC and the oceanic edge, $x = l$, along AD. It follows that $AB = CD = l = 380$ km. The depth h is taken as 100 m which is approximately the mean depth of water within $ABCD$. Using the relation: $\gamma = 2\Omega \sin\phi$, with $\Omega = 0{\cdot}2625$/h and $\phi = 50^\circ\,42'$, leads us to assume $\gamma = 0{\cdot}405$/h. Also we take $k = 0{\cdot}045$/h (Lauwerier 1957*a*) and $g = 9{\cdot}81$ m/s^2.

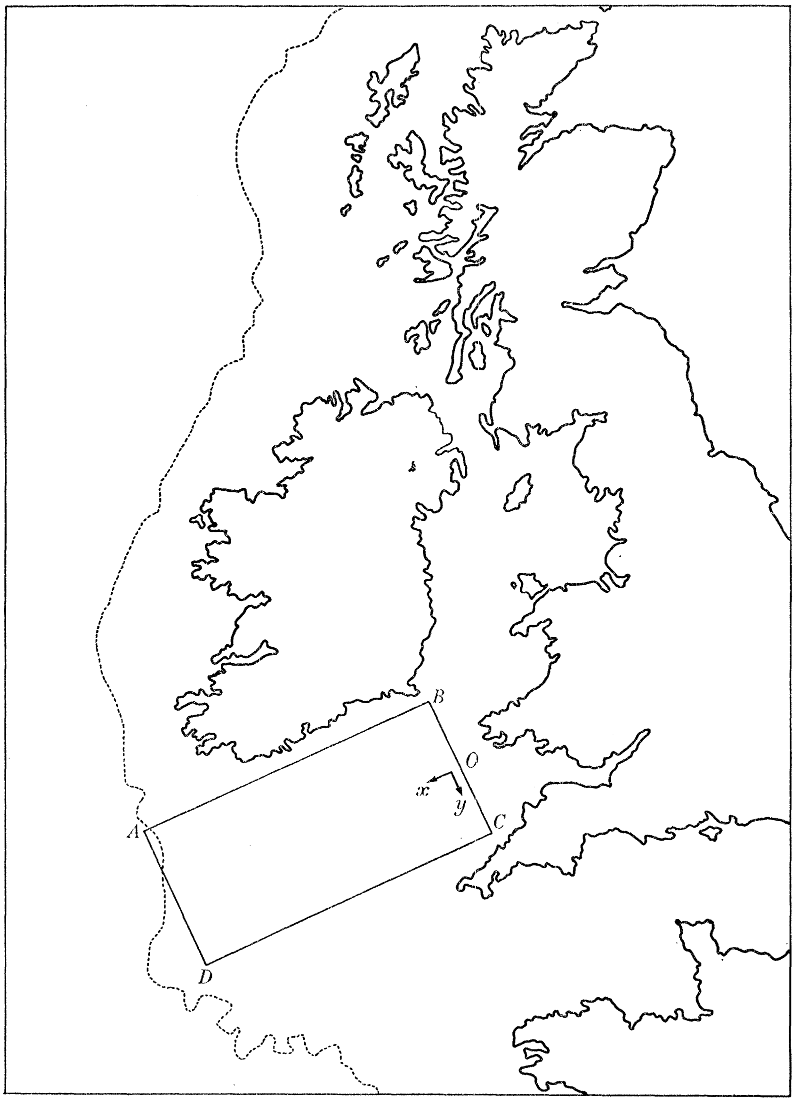

Figure 5. Shelf area ($ABCD$) considered in the numerical example; ---- 100 fathoms.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The first step in the numerical procedure is to evaluate the roots of equation (39). This has been done for $n = 1, 2, \ldots, 10$, and the values of λ_n, μ_n, ν_n obtained are given in table 1.

The subsequent numerical work is divided into two parts. First, using expressions derived in §7, the elevation of the sea surface along the coastal edge BC, produced by a surge input to the shelf along its oceanic edge AD, is determined: two cases are considered corresponding to an incoming surge of 9 h duration and one of 18 h duration, respectively. Secondly, using results obtained in §8 and §9, the response of the sea level along BC to wind fields acting over the area $ABCD$ is investigated.

Table 1. Values of λ_n, μ_n, ν_n (hour^{-1})

n	λ_n	μ_n	ν_n
1	0·051078	0·064461	0·61532
2	0·083032	0·048484	1·4550
3	0·087361	0·046319	2·3649
4	0·088634	0·045683	3·2874
5	0·089169	0·045416	4·2141
6	0·089442	0·045279	5·1428
7	0·089600	0·045200	6·0725
8	0·089699	0·045150	7·0029
9	0·089765	0·045117	7·9338
10	0·089812	0·045094	8·8649

(*a*) *Surge input from the ocean*

We consider the case when $\pi/\omega = 9$ h. Then, from (12) and (36),

$$\zeta = H(t)\,\zeta_m \sin(\pi t/9) \quad \text{at} \quad x = l, \tag{73·1}$$

where t is measured in hours. After evaluating the functions σ, ψ, ξ, η given by (53), (54), and the functions $d_n, \theta_n, D_n, E_n, \Theta_n, \chi_n$ (for $n = 1, 2, \ldots, 10$) given by (55), (56), it follows from (57) that, at $x = 0$,

$$\begin{aligned}\frac{\zeta}{\zeta_m} = {} & 0·75946 \sin(0·34907t) - 0·28030 \cos(0·34907t) \\ & + 0·07971 \exp(-0·051078t) + 0·59299 \cos(0·61532t + 1·2033) \exp(-0·064461t) \\ & - 0·00740 \exp(-0·083032t) - 0·09967 \cos(1·4550t + 1·4953) \exp(-0·048484t) \\ & + 0·00176 \exp(-0·087361t) + 0·03730 \cos(2·3649t + 1·5296) \exp(-0·046319t) \\ & - 0·00066 \exp(-0·088634t) - 0·01924 \cos(3·2874t + 1·5423) \exp(-0·045683t) \\ & + 0·00031 \exp(-0·089169t) + 0·01169 \cos(4·2141t + 1·5489) \exp(-0·045416t) \\ & - 0·00017 \exp(-0·089442t) - 0·00784 \cos(5·1428t + 1·5530) \exp(-0·045279t) \\ & + 0·00010 \exp(-0·089600t) + 0·00562 \cos(6·0725t + 1·5558) \exp(-0·045200t) \\ & - 0·00007 \exp(-0·089699t) - 0·00423 \cos(7·0029t + 1·5578) \exp(-0·045150t) \\ & + 0·00005 \exp(-0·089765t) + 0·00329 \cos(7·9338t + 1·5594) \exp(-0·045117t) \\ & - 0·00003 \exp(-0·089812t) - 0·00264 \cos(8·8649t + 1·5606) \exp(-0·045094t),\end{aligned} \tag{73·2}$$

terms corresponding to $n > 10$ being ignored. The first two terms in (73·2) represent the *forced* response of the sea level at $x = 0$ to the oscillation applied at $x = l$ defined by (73·1). The motion on the shelf is generated from a state of rest at $t = 0$ and transient oscillations in the form of shelf seiches are set up. These are represented in (73·2) by the terms involving products of cosine and exponential functions. The terms, in the order given above, correspond respectively to $n = 1, 2, 3, \ldots$, and each represents a mode of oscillation of period $2\pi/\nu_n$. The mode corresponding to $n = 1$ is predominant with a period of approximately 10 h. The modes decrease in period and amplitude with increasing n. The pure exponential terms in (73·2) are present because of the rotation of the Earth: their contribution to ζ/ζ_m at $x = 0$ is one which decreases steadily with time. In the above example this non-periodic part of ζ/ζ_m is small compared with the free oscillatory part.

Let ζ given by (73·1) be denoted by ζ_l and take

$$\zeta = \zeta_l(t) + \zeta_l(t-9) \quad \text{at} \quad x = l. \tag{74·1}$$

Then the elevation of the sea surface at the oceanic edge of the shelf varies with the time in the form of a half sine wave, rising from zero at $t = 0$, attaining a maximum value of ζ_m, and falling to zero again at $t = 9$. The variation represents a surge input to the shelf of duration 9 h. Corresponding to this input we have the response

$$\zeta/\zeta_m = H(t)\,Z(t) + H(t-9)\,Z(t-9) \quad \text{at} \quad x = 0, \tag{74·2}$$

where $Z(t)$ denotes ζ/ζ_m given by (73·2).

Both the surge input at $x = l$ (defined by (74·1)) and the resulting elevation of the sea surface at $x = 0$ (obtained from (74·2)) are plotted in figure 6 (*a*). It is of interest to note from the figure that there is no amplification of the surge in its passage across the shelf—this may be attributed to dispersion caused by the rotation of the earth and dissipation of energy by friction. In the response at $x = 0$ we may recognize damped oscillatory motion due to shelf seiches. The fundamental mode with a period of approximately 10 h is evident; the higher modes are present to a lesser degree and give the response curve its somewhat irregular appearance. The mean level in the oscillatory motion decreases steadily as the time increases, indicating pure exponential terms in the response

Calculations have also been carried out for a second case in which the surge input to the shelf is of 18 h duration. The results are shown in figure 6 (*b*). The surge height turns out to be smaller at $x = 0$ than at $x = l$. Damped oscillatory motion with a period of approximately 10 h is again evident at $x = 0$, but the pure exponential response is now more prominent than in the first case.

(*b*) *Surges generated on the shelf by wind fields moving onshore*

Analytical results obtained in §§ 8 and 9 are now applied to study changes in sea level produced by wind fields acting over the rectangular area of shelf *ABCD* shown in figure 5.

Using equation (68), we can determine sea-level variations at $x = 0$ (along BC in figure 5) due to uniform stationary wind stress fields suddenly created over the shelf between (i) $x = l$ and $x = \frac{4}{5}l$, (ii) $x = \frac{4}{5}l$ and $x = \frac{3}{5}l$, (iii) $x = \frac{3}{5}l$ and $x = \frac{2}{5}l$, (iv) $x = \frac{2}{5}l$ and $x = \frac{1}{5}l$, (v) $x = \frac{1}{5}l$ and $x = 0$ (figure 7). Consider five such fields, denoted by F_1, F_2, F_3, F_4, F_5, associated respectively with the areas (i), (ii), (iii), (iv), (v). Then, superimposing them with suitable time lags between the commencement of each, a wind area moving towards the coast may be simulated. If, for example, the fields are created in succession in the order

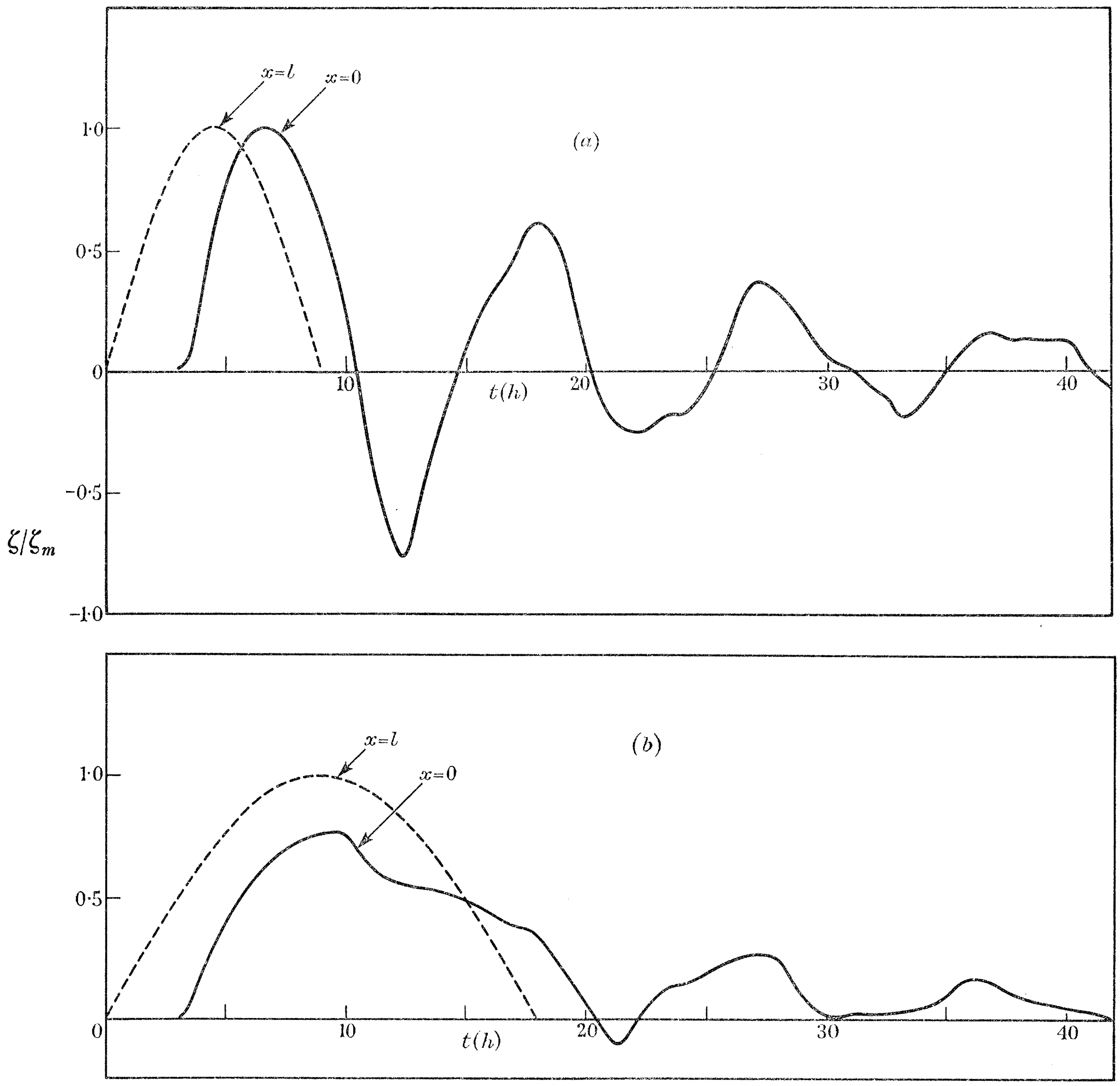

FIGURE 6*a*, *b*. Elevation of the sea surface at the coast ($x = 0$) in response to a rise and fall of the water level at the edge of the shelf ($x = l$) of duration (*a*) 9 h and (*b*) 18 h.

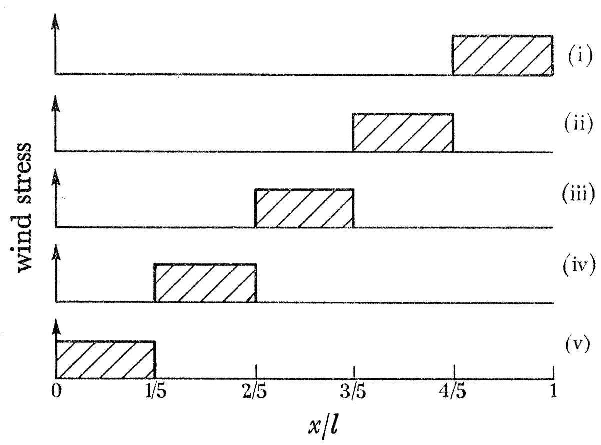

FIGURE 7. Diagram indicating uniform stationary wind stress fields over the shelf between (i) $x = l$ and $x = \frac{4}{5}l$; (ii) $x = \frac{4}{5}l$ and $x = \frac{3}{5}l$; (iii) $x = \frac{3}{5}l$ and $x = \frac{2}{5}l$; (iv) $x = \frac{2}{5}l$ and $x = \frac{1}{5}l$; (v) $x = \frac{1}{5}l$ and $x = 0$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

F_1, F_2, F_3, F_4, F_5 with a time difference of t_0 between the beginning of each, then the effect is that of a wind area which spreads across the shelf from the oceanic edge towards the coast in time $5t_0$ behind a boundary moving onshore with speed $l/5t_0$ ($= V_e$, say). If each field is removed at a time $4t_0$ after its creation then, effectively, a wind belt of width $\frac{4}{5}l$ moves onshore across the shelf as shown in figure 8. The sea-level variations at $x = 0$ produced by such changes are obtained by superimposing, at appropriate times, the values of sea-surface elevation for the five constituent wind fields.

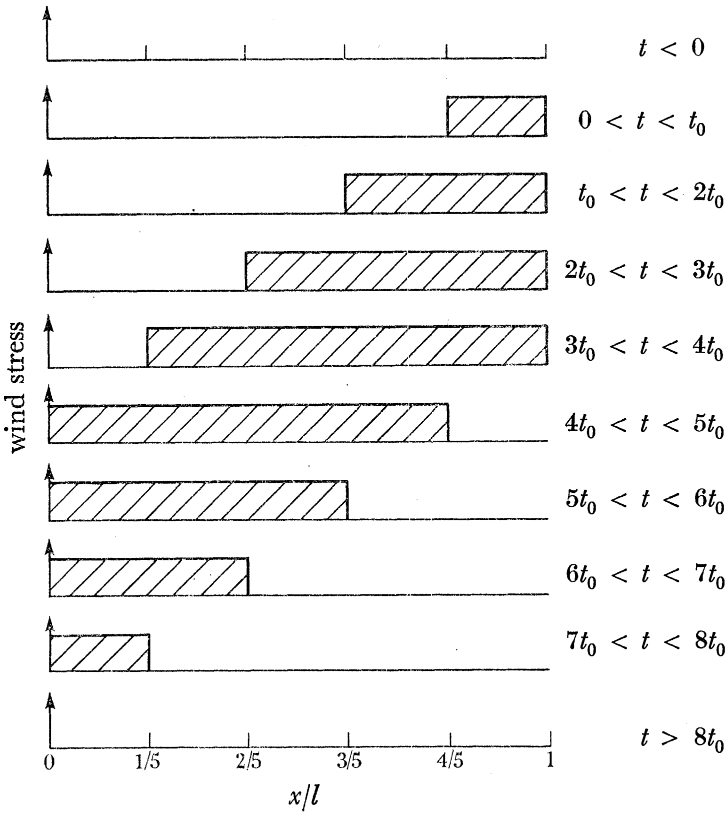

FIGURE 8. Wind stress field, of width $\frac{4}{5}l$, moving across the shelf towards the coast ($x = 0$) in steps. The effective speed of propagation of the field is $V_e = \frac{1}{5}l/t_0$.

First we deal with the case when the stress in each of the fields F_1, F_2, F_3, F_4, F_5 is of magnitude 1 dyn/cm^2, directed onshore. To determine the variation in elevation of the sea level at $x = 0$ produced by each field we take $P = 1$ dyn/cm^2, $Q = 0$ and $a = \frac{4}{5}l, \frac{3}{5}l, \frac{2}{5}l, \frac{1}{5}l, 0$ in equation (68). Expressions are obtained giving ζ_0 (ζ at $x = 0$) for each value of a. Each expression gives the response of the sea level at $x = 0$ due to a wind stress field of the type defined by equation (23) and shown in figure 2. Then, at $x = 0$: $\zeta = \zeta_0\ (a = \frac{4}{5}l)$ for field F_1,

$$\zeta = \zeta_0(a = \tfrac{3}{5}l) - \zeta_0(a = \tfrac{4}{5}l) \quad \text{for field } F_2,$$

$$\zeta = \zeta_0(a = \tfrac{2}{5}l) - \zeta_0(a = \tfrac{3}{5}l) \quad \text{for field } F_3,$$

$$\zeta = \zeta_0(a = \tfrac{1}{5}l) - \zeta_0(a = \tfrac{2}{5}l) \quad \text{for field } F_4,$$

$$\zeta = \zeta_0(a = 0) - \zeta_0(a = \tfrac{1}{5}l) \quad \text{for field } F_5.$$

As an example of the mathematical forms obtained from equation (68) we give $\zeta_0(a = \frac{4}{5}l)$: numerical work yields

$$\begin{aligned}\zeta_0(a = \tfrac{4}{5}l) &= 0{\cdot}024797\\ &-0{\cdot}013572\exp(-0{\cdot}051078t) - 0{\cdot}017733\cos(0{\cdot}61532t - 0{\cdot}16760)\exp(-0{\cdot}064461t)\\ &+0{\cdot}000700\exp(-0{\cdot}083032t) + 0{\cdot}008340\cos(1{\cdot}4550t - 0{\cdot}03810)\exp(-0{\cdot}048484t)\\ &-0{\cdot}000118\exp(-0{\cdot}087361t) - 0{\cdot}003903\cos(2{\cdot}3649t - 0{\cdot}02070)\exp(-0{\cdot}046319t)\\ &+0{\cdot}000025\exp(-0{\cdot}088634t) + 0{\cdot}001634\cos(3{\cdot}2874t - 0{\cdot}01431)\exp(-0{\cdot}045683t)\\ &-0{\cdot}000003\exp(-0{\cdot}089169t) - 0{\cdot}000380\cos(4{\cdot}2141t - 0{\cdot}01097)\exp(-0{\cdot}045416t)\\ &-0{\cdot}000002\exp(-0{\cdot}089442t) - 0{\cdot}000255\cos(5{\cdot}1428t - 0{\cdot}00892)\exp(-0{\cdot}045279t)\\ &+0{\cdot}000002\exp(-0{\cdot}089600t) + 0{\cdot}000479\cos(6{\cdot}0725t - 0{\cdot}00751)\exp(-0{\cdot}045200t)\\ &-0{\cdot}000001\exp(-0{\cdot}089699t) - 0{\cdot}000445\cos(7{\cdot}0029t - 0{\cdot}00649)\exp(-0{\cdot}045150t)\\ &+0{\cdot}000001\exp(-0{\cdot}089765t) + 0{\cdot}000281\cos(7{\cdot}9338t - 0{\cdot}00572)\exp(-0{\cdot}045117t)\\ &\qquad - 0{\cdot}000086\cos(8{\cdot}8649t - 0{\cdot}00511)\exp(-0{\cdot}045094t)\end{aligned} \tag{75}$$

terms corresponding to $n > 10$ being ignored. Here ζ_0 is measured in feet and t in hours. The first term of (75) gives the elevation in the steady state maintained by the wind; the remaining terms, representing the transient part of the response, are similar in form to those of (73·2).

Having determined the sea-level response at $x = 0$ to each of the fields F_1 to F_5, we are now in a position to superimpose these fields, in the manner already described, and obtain the elevation of the sea surface at $x = 0$ produced by a belt of onshore wind of width $\frac{4}{5}l$ (exerting a stress of 1 dyn/cm²) which moves across the shelf towards the coast with an effective speed $V_e(= l/5t_0)$ as indicated in figure 8. The results of this superposition are shown in figure 9(*a* to *f*). The following six cases, corresponding to different speeds of propagation of the wind field, are considered:

$V_e = 82$ kt $(t_0 = 0{\cdot}5\text{ h})$, figure 9(*a*);
$V_e = 41$ kt $(t_0 = 1\text{ h})$, figure 9(*b*);
$V_e = 27\frac{1}{3}$ kt $(t_0 = 1{\cdot}5\text{ h})$, figure 9(*c*);
$V_e = 20\frac{1}{2}$ kt $(t_0 = 2\text{ h})$, figure 9(*d*);
$V_e = 16\frac{2}{5}$ kt $(t_0 = 2{\cdot}5\text{ h})$, figure 9(*e*);
$V_e = 13\frac{2}{3}$ kt $(t_0 = 3\text{ h})$, figure 9(*f*).

The diagrams show that the peak value of sea-surface elevation does not vary appreciably with the effective speed of propagation, V_e. However, the duration of the surge associated with this peak value increases as V_e decreases. The response curves for the higher values of V_e (figure 9(*a*), (*b*), (*c*)) are characterized by a damped oscillation with a period of approximately 10 h, arising from the excitation of the fundamental shelf seiche. For the smaller value of V_e (figures 9(*d*), (*e*), (*f*)) an exponential decay of the peak elevation may be recognized. Short-period waves, superimposed upon the main response, are evident in all six cases: they are particularly noticeable in figures 9(*a*), (*f*). These are due to the presence of higher order shelf seiches, and also arise because of the discontinuous movement (in steps) of the wind field across the shelf.

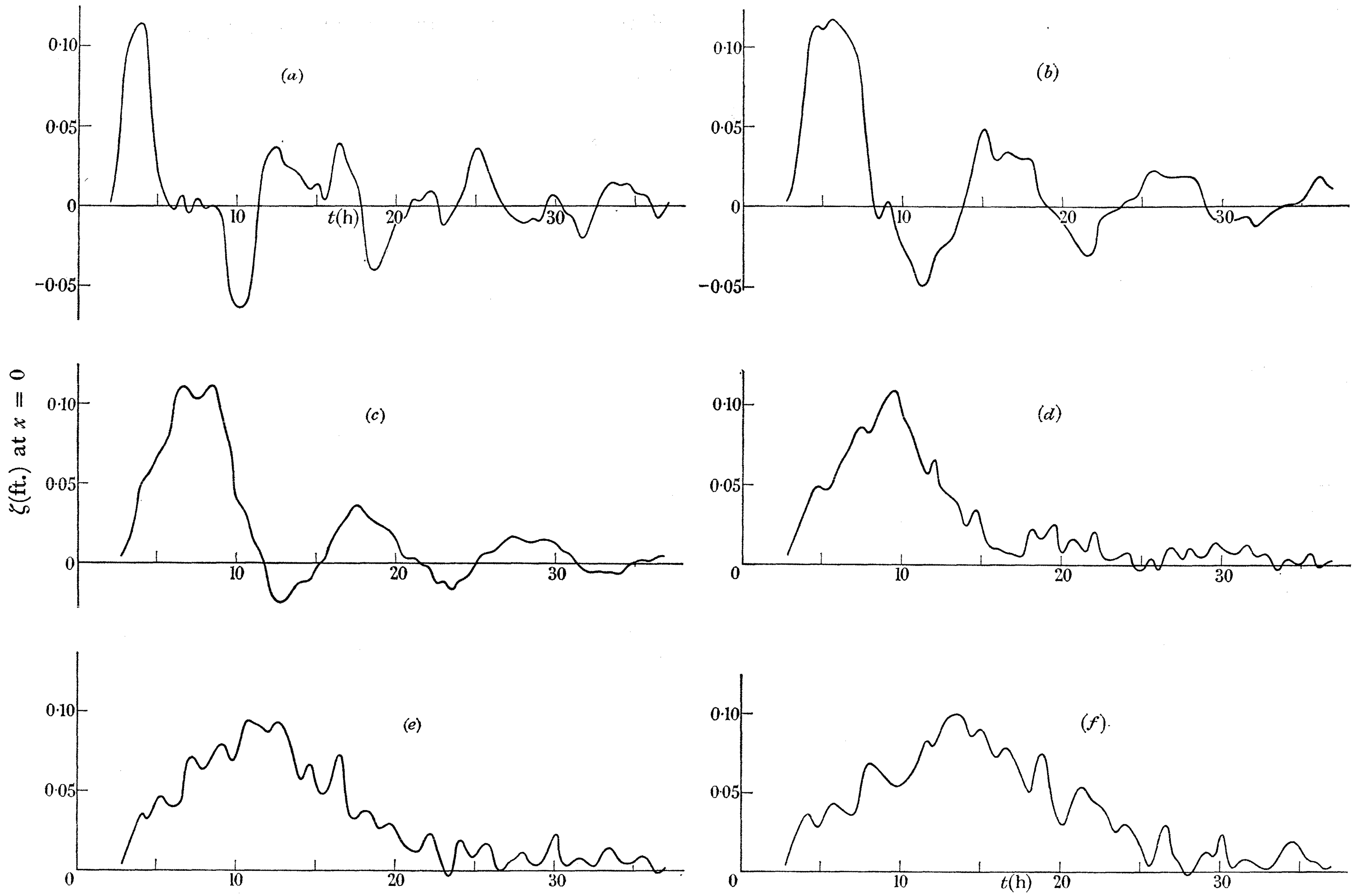

FIGURE 9. Surge due to an onshore wind stress field of 1 dyn/cm² and width $\frac{4}{5}l$, moving across the shelf towards the coast in steps:

	(a)	(b)	(c)	(d)	(e)	(f)
t_0(h)	0·5	1·0	1·5	2·0	2·5	3·0
V_e(kt)	82	41	$27\frac{1}{3}$	$20\frac{1}{2}$	$16\frac{2}{5}$	$13\frac{2}{3}$

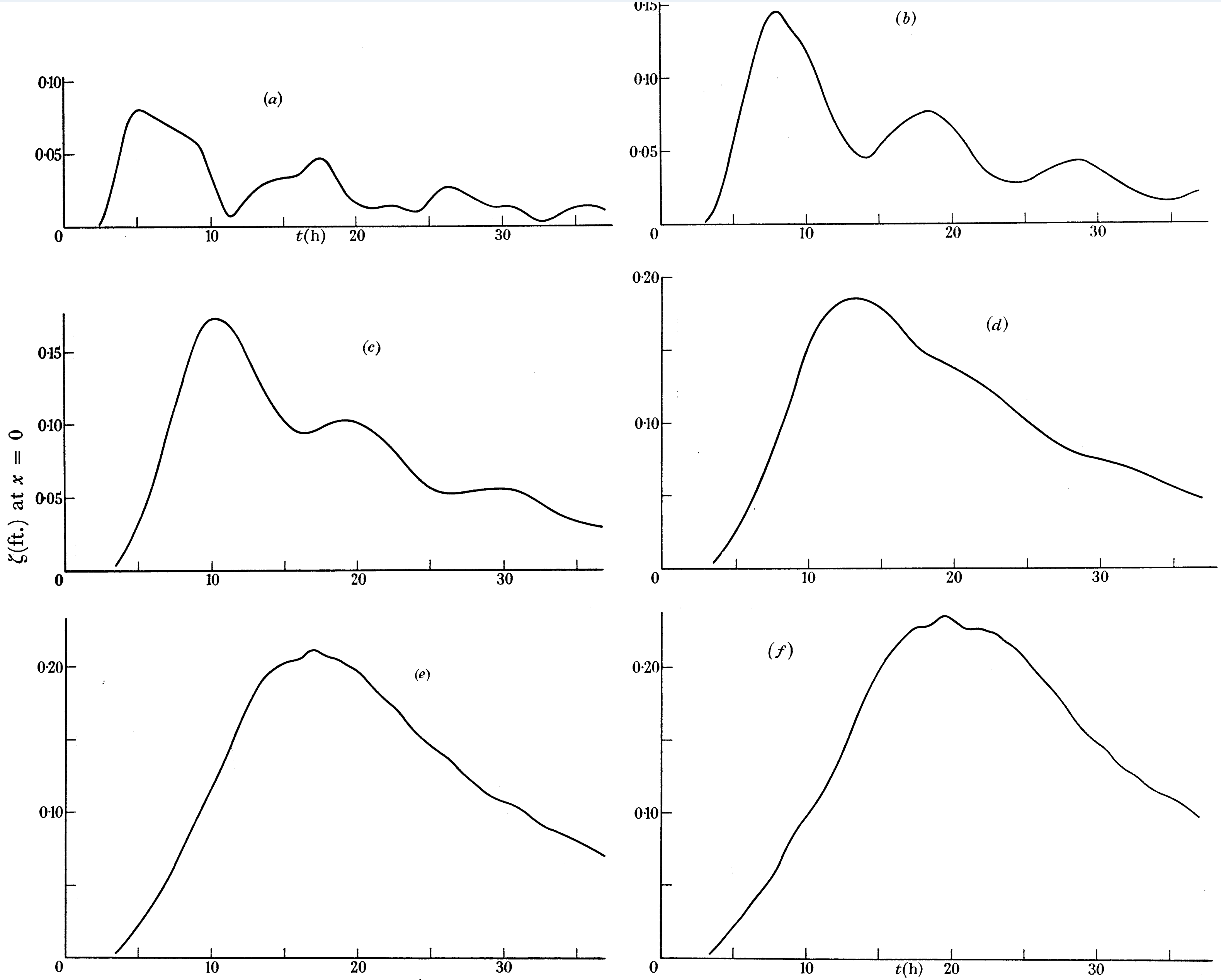

FIGURE 10. Surge due to a longshore wind stress field of 1 dyn/cm² and width $\frac{4}{5}l$, moving across the shelf towards the coast in steps:

	(a)	(b)	(c)	(d)	(e)	(f)
t_0(h)	0·5	1·0	1·5	2·0	2·5	3·0
V_e(kt)	82	41	$27\frac{1}{3}$	$20\frac{1}{2}$	$16\frac{2}{5}$	$13\frac{2}{3}$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Next, we investigate the response of the sea level at $x = 0$ to a belt of *longshore* wind which moves across the shelf towards the coast. The coastline $x = 0$ lies to the right of an observer looking in the direction of the wind. The procedure described above is repeated taking $P = 0$, $Q = -1$ dyn/cm². The results are shown in figure 10 (a to f). The wind belt is again of width $\frac{4}{5}l$ and the values of V_e considered are the same as before. The diagrams show that, in response to the wind, the level of the sea surface rises to a maximum and then falls gradually to zero: for the higher values of V_e the decay of the elevation is accompanied by oscillations of approximate period ten hours. The lower the speed of propagation of the wind field, the greater is the peak value of elevation. A comparison of figures 9 and 10 indicates that the surge produced by the longshore wind field is greater than that produced by the

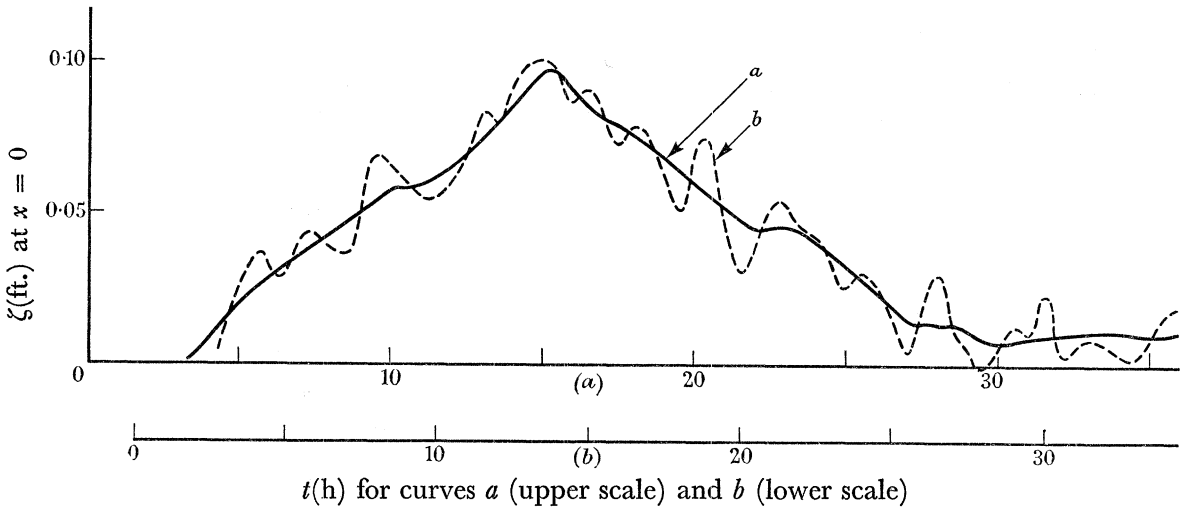

FIGURE 11. Surge due to an onshore wind stress field of 1 dyn/cm² and width $\frac{4}{5}l$, moving across the shelf towards the coast (a) continuously: $V = 13\frac{2}{3}$ kt, and (b) in steps: $t_0 = 3$ h, $V_e = 13\frac{2}{3}$ kt.

corresponding onshore field, providing that the speed V_e is small enough, i.e. providing that the wind acts for a long enough time over the shelf water. In the generation of a surge by longshore wind the current set up in the direction of the wind is, in the northern hemisphere, deflected to the right under the influence of the Earth's rotation: when the coast lies to the right of an observer looking in the direction of the wind (as in the present example), the onshore current thus induced produces a rise in elevation at the coast (a positive surge); when the coast lies to the left of an observer looking in the direction of the wind, the off-shore current which is induced produces a lowering of the sea level at the coast (a negative surge).

So far, we have studied the effects on the sea level at $x = 0$ of a wind belt moving onshore in a series of steps, each step being separated by a constant time interval, t_0. Alternatively, using equations (71·1), (71·2), we can determine the sea-level variations due to a similar wind area which moves across the shelf *continuously* with a uniform speed V. The case of a wind belt of width $\frac{4}{5}l$, with $P = 1$ dyn/cm², $Q = 0$ (corresponding to an onshore wind stress of 1 dyn/cm²) and $V = 13\frac{2}{3}$ kt, is investigated. The response of the sea surface at $x = 0$ to this continuously moving wind field is shown in figure 11. The response to the same field when it moves in steps towards the coast, with effective speed: $V_e = 13\frac{2}{3}$ kt (figure 9 (f)), is also shown in figure 11. There is good agreement between the two response curves—allowing a time-lag of $1\frac{1}{2}(=\frac{1}{2}t_0)$ h between them in order to bring the distance–time graphs for the two

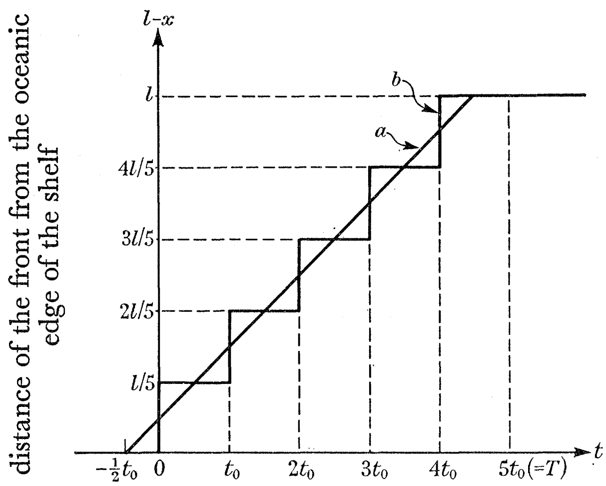

Figure 12. Distance–time graphs for the propagation of a wind-field front across the shelf: (a) front with steady speed $V = l/T$; (b) front moving in steps.

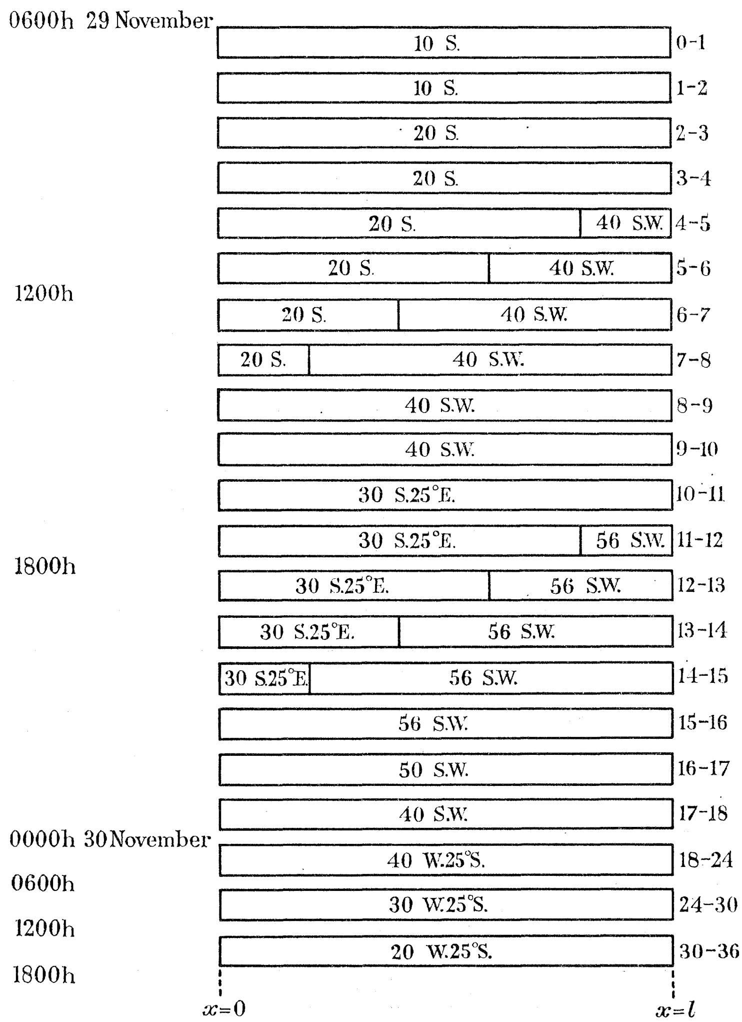

Figure 13. Theoretical model of the wind fields over the shelf $ABCD$ from 06.00 h G.M.T. 29 November to 18.00 h G.M.T. 30 November 1954. Each rectangle represents the shelf area with its oceanic edge ($x = l$) lying to the right, and shows the wind speed in knots, followed by the wind direction, during a given time interval.

modes of propagation into correspondence (figure 12). Clearly, the irregularities superimposed upon the main curve in figure 9 (*f*) are chiefly caused by the discontinuous movement of the wind field across the shelf. The response to the wind field, when it moves uniformly across the shelf towards the coast with a speed $V = 41$ kt, has also been calculated. The result obtained agrees very closely with the response to the field when it moves in steps towards the coast with effective speed $V_e = 41$ kt (figure 9 (*b*)). This shows, as expected, that the response to a field moving onshore in steps agrees more nearly with the response to a continuously moving field as the time-step is made shorter.

(*c*) *Surges at Milford Haven*

Figure 13 shows a theoretical model of the wind fields over the shelf area *ABCD* (figure 5), from 06.00 h 29 November to 18.00 h 30 November 1954, constructed from information given by the *Daily Weather Report* of the Meteorological Office. A surge of approximately 6 ft. was observed at Milford Haven at 02.00 h 30 November (figure 15). This port lies near to the midpoint of side *BC* ($x = 0$) of the sea area *ABCD*, and it is therefore of interest to know whether elevation of the sea surface at $x = 0$ obtained from the theory on the basis of the model shown in figure 13 is able to account for the surge.

The significant meteorological changes over the area *ABCD* during the period considered may be briefly summarized as follows. Southerly winds (10 to 20 kt) are displaced by S.W. winds (40 kt) which extend across the area behind a warm front and later back to S 25° E. A cold front subsequently moves across the area followed by high S.W. winds (56 kt) which later veer to the west. The timing of these events is indicated in figure 13. The fronts are associated with a depression, the centre of which moves along the Irish Coast (figure 14).

The wind speeds indicated in figure 13 are converted to wind stresses by means of the formula

$$\tau_w = c\rho_a U^2,$$

where τ_w is the wind stress, U the wind speed, ρ_a the density of the air (taken as 0·00125 g/cm³), and c the drag-coefficient. We take $c \times 10^3 = -0{\cdot}12 + 0{\cdot}137U$ for $5 < U < 19{\cdot}22$ (Professor P. A. Sheppard—private communication—drag inferred from wind measurements by application of the well-known logarithmic 'law of the wall') and $c \times 10^3 = 2{\cdot}513$ for $U \geqslant 19{\cdot}22$ (Charnock & Crease 1957) where U is measured in m/sec. Table 2 gives wind stress in dynes/cm² obtained for the various values of wind speed occurring in figure 13.

The changing pattern of wind stress over the area *ABCD*, thus deduced, is expressed as a combination of onshore and longshore stress fields, each of magnitude 1 dyn/cm², of the five types: F_1, F_2, F_3, F_4, F_5 (figure 7). The response of the sea level at $x = 0$ to each field is known: values of sea-surface elevation associated with each such response are superimposed, at appropriate times, to obtain the sea-level variations due to the combination of fields. In this way the elevation of the sea surface at $x = 0$ resulting from the sequence of changes in wind speed and direction given by figure 13 is determined. The result, after the elevation due to changes in barometric pressure at the midpoint of *BC* has been added to it, is shown in figure 15. The curve obtained, giving the surge at the midpoint of *BC* due to the effects of both wind and pressure, agrees satisfactorily with the observed surge at Milford Haven. The first peak in elevation at 16.00 h may be attributed to the passage of the first front and the wind field following it. The second higher peak at 00.30 h is mainly due to the strong winds following the second front, but the oscillatory motion set up by the first wind field also contributes to the high value. The timing of the passage of the fronts over the shelf water is an important factor in the generation of the surge.

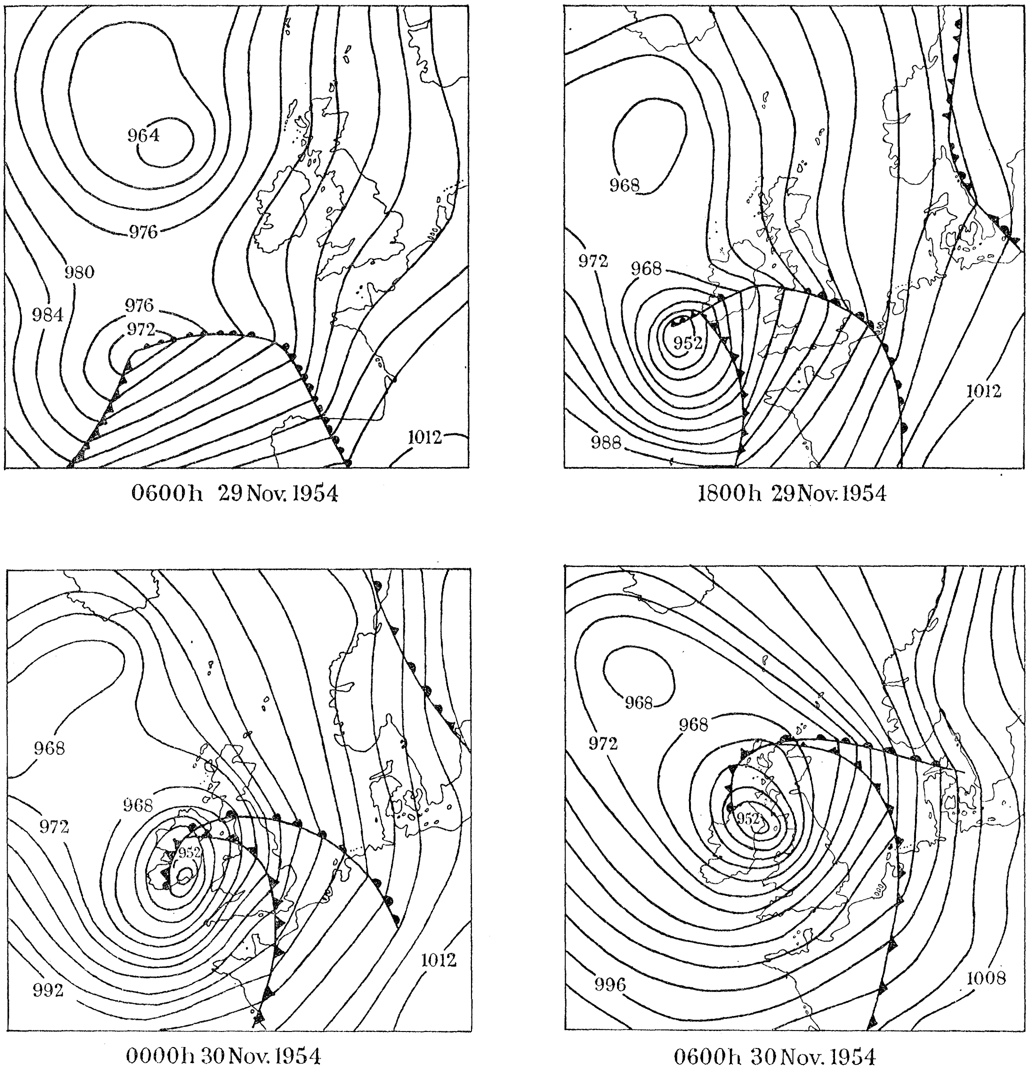

FIGURE 14. Synoptic charts, with an isobar interval of 4 mb.

TABLE 2. WIND STRESS CORRESPONDING TO VARIOUS VALUES OF WIND SPEED

wind speed (kt)	wind stress (dyn/cm²)
10	0·2
20	1·7
30	5·9
40	13·3
50	20·8
56	26·1

The statical law (Charnock & Crease 1957) is used to calculate the elevation of the sea surface due to changes in barometric pressure. Dynamical effects in the sea associated with barometric pressure variations are, therefore, neglected. This neglect may be partly responsible for the differences between the calculated and observed surges shown in figure 15. Differences between them are, however, expected since the theoretical surge applies to a

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

position some distance offshore from Milford Haven. It is hoped to investigate the dynamical effects of barometric pressure in a later paper.

In a further example, an attempt is made to reproduce theoretically the surge of approximately 2 ft observed at Milford Haven at 07.00 h 21 September 1953. A model of the wind

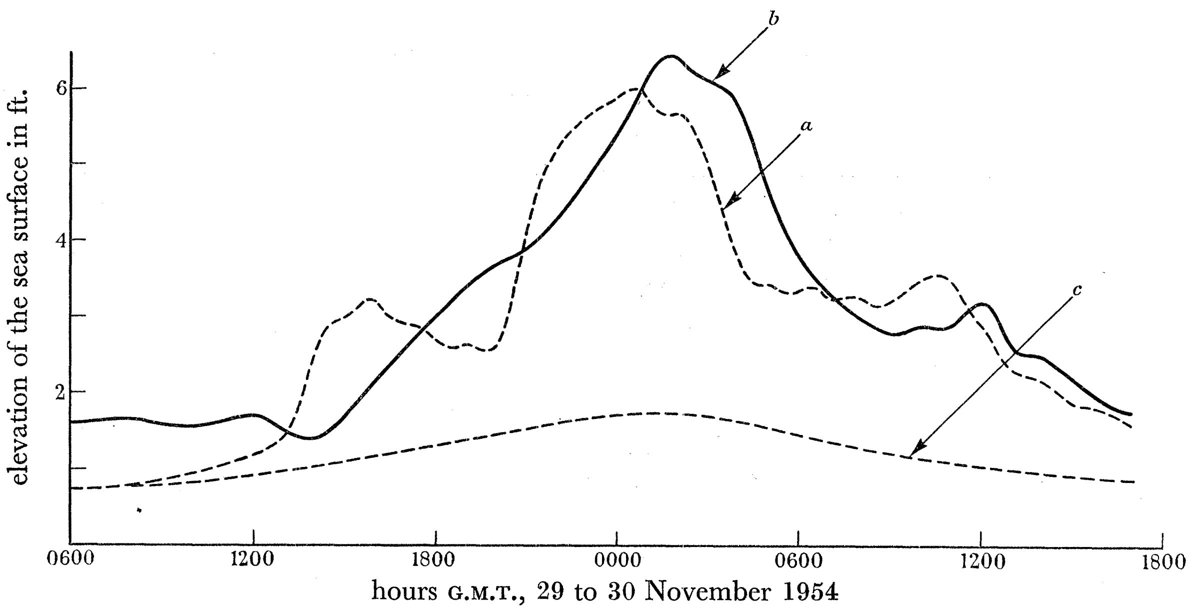

FIGURE 15. Surge at $x = 0$ obtained from the present theory (*a*), compared with the observed surge at Milford Haven (*b*). The theoretical surge consists of a part due to wind stress, and a part (*c*) due to changes in barometric pressure.

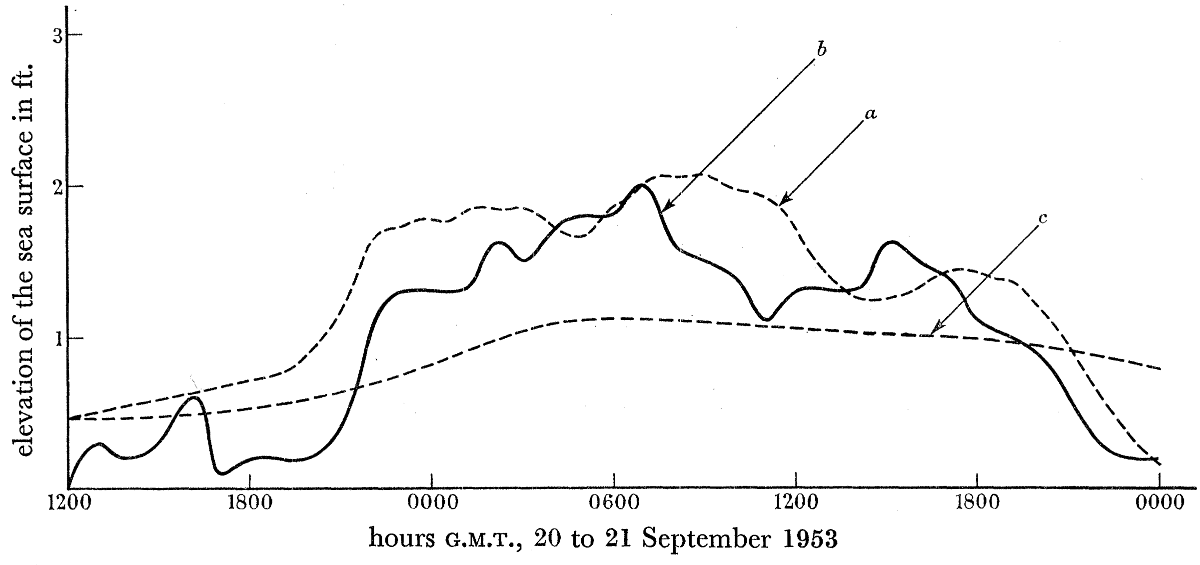

FIGURE 16. Surge at $x = 0$ obtained from the present theory (*a*), compared with the observed surge at Milford Haven (*b*); (*c*) the surge due to changes in barometric pressure.

fields over the shelf area *ABCD* from 12.00 h 20 September to 00.00 h 22 September, similar in form to that given in figure 13, is constructed. From this we deduce the wind surge at $x = 0$, to which is added a contribution due to changes in barometric pressure. The procedure is, in fact, the same as that described in the first example and the results obtained are shown in figure 16. The theory predicts a surge of 2·06 ft. occurring between 07.00 and

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

09.00 h, 21 September, which agrees satisfactorily with the observed surge. The wind effect on the sea level is much smaller than that in the first example—owing mainly to lighter winds. Also, in the present case, only one meteorological front (followed by its associated wind field) passes over the shelf area *ABCD*, whereas in the first case there is the combined effect of two fronts.

The meteorological changes over the area *ABCD* from 12.00 h 20 September to 00.00 h 22 September 1953 (relevant to the last example) were largely determined by a depression which moved eastwards across the British Isles. After moving in from the Atlantic, its centre crossed Northern Ireland during the interval 06.00 to 12.00 h, 21 September, and at 00.00 h 22 September was located near the border between England and Scotland. A front associated with the depression moved across the sea area *ABCD* between 18.00 h 20 September and 00.00 h 21 September. This displaced westerly winds (18 kt)—which progressively backed to the south and increased in strength to 28 kt—by stronger westerlies (30 to 33 kt) which veered to W 30° N as the depression moved away to the east.

11. Conclusions

A theory has been developed for the calculation of surge levels, along an open coast, associated with the onshore movement of wind fields across a shelf sea which adjoins the coast. Application of the theory indicates that surges generated in this way occur on the west coast of the British Isles. In such cases the wind fields, associated with moving depressions, act over the sea area to the south of Ireland.

The character of the surge along an open coast produced by a wind field moving onshore is influenced considerably by the speed of propagation of the field. For conditions appropriate to the west coast of the British Isles, results obtained from the theory show that surges corresponding to speeds of propagation above about 20 kt exhibit oscillations due to the excitation of shelf seiches; for lower speeds, however, the surges are predominantly aperiodic in type. Examples worked out in this paper show the possibility of greater surges being generated by fields of longshore wind than by the corresponding fields of onshore wind.

From a theoretical investigation we conclude that oceanic surges incident on the continental shelf to the south of Ireland are not amplified in their passage across the shelf to the coast—due to dispersion caused by the rotation of the Earth, and dissipation of energy by friction. To test the theory in this respect, observations of sea level at the oceanic edge of the shelf would be required, in addition to those at coastal positions.

References

Charnock, H. & Crease, J. 1957 *Sci. Progr.* **45**, 494–511.
Crease, J. 1956 *Proc. Roy. Soc.* A, **233**, 556–569.
Freeman, J. C., Jr., Baer, L. & Jung, C. H. 1957 *J. Mar. Res.* **16**, 12–22.
Goldsbrough, G. R. 1952 *Mon. Not. R. Astr. Soc. Geophys. Suppl.* **6**, 365–371.
Groen, P. & Groves, G. W. 1962 *The sea*, **1**, 611–646. New York: Interscience Publishers.
Hidaka, K. 1953 *A. and M. Coll. of Texas, Dep. Oceanogr. and Met., Tech. Rep.* no. 9.
Ichiye, T. 1950 *Mem. Kobe Mar. Obs.* **8**, 19–25.
Ichiye, T. 1962 *Nat. Hurricane Res. Proj. Rep.* no. 50, U.S. Weather Bur. pp. 255–266.
Jeffreys, H. 1923 *Phil. Mag.* **46**, 114–125.
Kajiura, K. 1956 *A. and M. Coll. of Texas, Dep. Oceanogr. and Met., Tech. Rep.* ref. 56-26T.

Kajiura, K. 1959 *A. and M. Coll. of Texas, Dep. Oceanogr. and Met., Tech. Rep.* ref. 59-23F.
Lauwerier, H. A. 1955 *Math. Cent. Amsterdam, Rep.* TW 32.
Lauwerier, H. A. 1957*a* *Math. Cent. Amsterdam, Rep.* TW 41.
Lauwerier, H. A. 1957*b* *Math. Cent. Amsterdam, Rep.* TW 42.
Lauwerier, H. A. 1962 *Math. Cent. Amsterdam, Rep.* TW 79.
Lennon, G. W. 1963 *Quart. J. R. Met. Soc.* **89**, 381–394.
Miyazaki, M. 1952 *Oceanogr. Mag.* **4** (1), 1–11.
Miyazaki, M. 1956 *Geophys. Mag., Tokyo*, **27** (3), 401–449.
Nomitsu, T. 1934 *Mem. Coll. Sci. Kyoto*, A, **17**, 249–280.
Nomitsu, T. 1935 *Mem. Coll. Sci. Kyoto*, A, **18**, 201–214.
Nomitsu, T. & Takegami, T. 1934 *Mem. Coll. Sci. Kyoto*, A, **17**, 93–141.
Phillips, E. G. 1945 *Functions of a complex variable with applications.* Edinburgh: Oliver and Boyd.
Proudman, J. 1953 *Dynamical oceanography.* London: Methuen and Co. Ltd.
Proudman, J. 1954*a* *Arch. Meteorol. Geophys. Bioklim.* A, **7**, 344–351.
Proudman, J. 1954*b* *Mon. Not. R. Astr. Soc. Geophys. Suppl.* **7**, 44–48.
Reid, R. O. 1956 *U.S. Army Beach Erosion Bd. Tech. Mem.* no. 83.
Takegami, T. 1936 *Mem. Coll. Sci. Kyoto*, A, **19**, 109–130.
Welander, P. 1961 *Advance. Geophys.* **8**, 316–379. New York and London: Academic Press.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES